

ON INTEGRABILITY OF HIROTA-KIMURA TYPE DISCRETIZATIONS

MATTEO PETRERA, ANDREAS PFADLER, AND YURI B. SURIS

ABSTRACT. We give an overview of the integrability of the Hirota-Kimura discretization method applied to algebraically completely integrable (a.c.i.) systems with quadratic vector fields. Along with the description of the basic mechanism of integrability (Hirota-Kimura bases), we provide the reader with a fairly complete list of the currently available results for concrete a.c.i. systems.

CONTENTS

1. Introduction	2
2. Some general results on invariant measures	4
3. Hirota-Kimura bases and integrals	5
4. Weierstrass differential equation	8
5. Some two-dimensional integrable systems	10
5.1. The three-dimensional Suslov system	10
5.2. Reduced Nahm equations	11
6. Euler top	12
7. Zhukovski-Volterra system	17
7.1. ZV system with two vanishing β_k 's	18
7.2. ZV system with one vanishing β_k	19
7.3. ZV system with all β_k 's non-vanishing	20
8. Volterra chain	20
8.1. Periodic Volterra chain with $N = 3$ particles	20
8.2. Periodic Volterra chain with $N = 4$ particles	21
9. Dressing chain ($N = 3$)	23
10. Coupled Euler tops	25
11. Three wave system	28
12. Lagrange top	30
13. Kirchhoff case of the rigid body motion in an ideal fluid	34
14. Clebsch case of the rigid body motion in an ideal fluid	37
14.1. First flow of the Clebsch system	38
14.2. General flow of the Clebsch system	41
15. $\mathfrak{su}(2)$ rational Gaudin system with $N = 2$ spins	43
16. Conclusions	46
References	47

Date: February 7, 2011.

Institut für Mathematik, MA 7-2, Technische Universität Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany.

E-mail: petrera@math.tu-berlin.de, pfadler@math.tu-berlin.de, suris@math.tu-berlin.de.

1. INTRODUCTION

The discretization method studied in this paper seems to be introduced in the geometric integration literature by W. Kahan in the unpublished notes [10]. It is applicable to any system of ordinary differential equations for $x : \mathbb{R} \rightarrow \mathbb{R}^n$ with a quadratic vector field:

$$\dot{x} = Q(x) + Bx + c,$$

where each component of $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a quadratic form, while $B \in \text{Mat}_{n \times n}$ and $c \in \mathbb{R}^n$. Kahan's discretization reads as

$$\frac{\tilde{x} - x}{\epsilon} = Q(x, \tilde{x}) + \frac{1}{2}B(x + \tilde{x}) + c, \quad (1)$$

where

$$Q(x, \tilde{x}) = \frac{1}{2}(Q(x + \tilde{x}) - Q(x) - Q(\tilde{x}))$$

is the symmetric bilinear form corresponding to the quadratic form Q . Here and below we use the following notational convention which will allow us to omit a lot of indices: for a sequence $x : \mathbb{Z} \rightarrow \mathbb{R}$ we write x for x_k and \tilde{x} for x_{k+1} . Eq. (1) is *linear* with respect to \tilde{x} and therefore defines a *rational* map $\tilde{x} = f(x, \epsilon)$. Clearly, this map approximates the time- ϵ -shift along the solutions of the original differential system, so that $x_k \approx x(k\epsilon)$. (Sometimes it will be more convenient to use 2ϵ for the time step, in order to avoid appearance of various powers of 2 in numerous formulas.) Since eq. (1) remains invariant under the interchange $x \leftrightarrow \tilde{x}$ with the simultaneous sign inversion $\epsilon \mapsto -\epsilon$, one has the *reversibility* property

$$f^{-1}(x, \epsilon) = f(x, -\epsilon).$$

In particular, the map f is *birational*.

W. Kahan applied this discretization scheme to the famous Lotka-Volterra system and showed that in this case it possesses a very remarkable non-spiralling property. This example is briefly discussed in [17]. Some further applications of this discretization have been explored in [11].

The next, even more intriguing appearance of this discretization was in the two papers by R. Hirota and K. Kimura who (being apparently unaware of the work by Kahan) applied it to two famous *integrable* system of classical mechanics, the Euler top and the Lagrange top [8, 12]. For the purposes of the present text, integrability of a dynamical system is synonymous with the existence of a sufficient number of functionally independent conserved quantities, or integrals of motion, that is, functions constant along the orbits. We leave aside other aspects of the multi-faceted notion of integrability, such as Hamiltonian ones or explicit solutions. Surprisingly, the Kahan-Hirota-Kimura discretization scheme produced in both the Euler and the Lagrange cases of the rigid body motion *integrable* maps. Even more surprisingly, the mechanism which assures integrability in these two cases seems to be rather different from the majority of examples known in the area of integrable discretizations, and, more generally, integrable maps, cf. [25]. The case of the discrete time Euler top is relatively simple, and the proof of its integrability given in [8] is rather straightforward and easy to verify by hands. As it often happens, no explanation was given in [8] about how this result has been *discovered*. The “derivation” of integrals of motion for the discrete time Lagrange top in [12] is rather cryptic and almost uncomprehensible.

We use the term “Hirota-Kimura (HK) type discretization” for Kahan discretization in the context of integrable systems. At the Oberwolfach meeting on Geometric Integration in 2006, T. Ratiu proposed to apply the Hirota-Kimura discretization to the Clebsch case of the rigid body motion in an ideal fluid and to the Kovalevsky top. The claim on the integrability of the HK discretization of the Clebsch case was proven only several years later in [17], while the integrability of the HK discretization of the Kovalevsky top remains an open problem (although there are some indications in favor of its non-integrability). Anyway, the general question of integrability of the HK type discretizations turns out to be very intriguing and rather difficult. In the present overview, we will present a rather long list of examples of integrable HK discretizations. Actually, this list is so impressive that in [17] we conjectured that HK discretizations of algebraically integrable systems always remain integrable. At present, we have some indications that this conjecture is wrong (although a rigorous proof of non-integrability for any of the apparently non-integrable cases remains elusive). Nevertheless, the sheer length of our list of examples clearly shows that there exist some general mechanisms that ensure integrability at least under certain additional assumptions. We think that to uncover general structures behind the integrability of HK discretizations is a big and important challenge for the modern theory of (algebraically) integrable systems.

The structure of our overview is as follows. In Section 2 we demonstrate some general sufficient conditions for a HK discretization to be measure preserving. These conditions are not related to integrability and do not cover all special cases considered in the main text. All our examples turn out to be measure preserving but for the majority of them we only can prove this property individually and do not know any general mechanisms. In Section 3 we present a formalization of the HK mechanism from [12], which will hopefully unveil its main idea and contribute towards demystifying at least some of its aspects. We introduce a notion of a “Hirota-Kimura basis” (HK basis) for a given map f . Such a basis Φ is a set of simple (often monomial) functions, $\Phi = (\varphi_1, \dots, \varphi_l)$, such that for every orbit $\{f^i(x)\}$ of the map f there is a certain linear combination $c_1\varphi_1 + \dots + c_l\varphi_l$ of functions from Φ vanishing on this orbit. As explained in Sect. 3, this is a *new* mathematical notion, not reducible to that of integrals of motion, although closely related to the latter. We lay a theoretical fundament for the search for HK bases for a given discrete time system, and discuss some practical recipes and tricks for doing this. Sections 4–15 contain our list of examples of algebraically integrable systems with quadratic vector fields, for which the HK discretization preserves integrability. For all the examples we provide the reader with a rather complete set of currently available results. The proofs are omitted almost everywhere. One of the reasons is that our investigations are based mainly on computer experiments, which are used both for *discovery* of new results and for their rigorous *proof*. For a given system, a search for HK bases can be done with the help of *numerical experiments*. If the search has been successful and a certain set of functions Φ has been identified as a HK basis for a given map f , then numerical experiments can provide a very convincing evidence in favor of such a statement. A rigorous proof of such a statement turns out to be much more demanding. At present, we are not in possession of any theoretical proof strategies and are forced to verify the corresponding statements by means of *symbolic computations* which in some cases turn out to be hardly feasible due to complexity issues. All these issues are intentionally avoided in our presentation here; an interested reader may consult [17] for a detailed discussion of one concrete example (Clebsch system). Our main goal here is to document the available results on integrable

HK discretizations and to attract attention of specialists in integrable systems and in algebraic geometry to these beautiful and mysterious objects which are definitely worth further investigation.

2. SOME GENERAL RESULTS ON INVARIANT MEASURES

In this section, we prove the existence of invariant measures for Kahan discretizations of two classes of dynamical systems with quadratic vector fields (not necessarily integrable).

The first class reads:

$$\dot{x}_i = \sum_{j=1}^N a_{ij} x_j^2 + c_i, \quad 1 \leq i \leq N,$$

with a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^N = -A^T$. The Kahan's discretization reads:

$$\tilde{x}_i - x_i = \epsilon \sum_{j=1}^N a_{ij} x_j \tilde{x}_j + \epsilon c_i, \quad 1 \leq i \leq N. \quad (2)$$

Proposition 1. *The map $\tilde{x} = f(x, \epsilon)$ defined by eqs. (2) has an invariant volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\phi(\tilde{x}, \epsilon)}{\phi(x, \epsilon)} \Leftrightarrow f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge \dots \wedge dx_N}{\phi(x, \epsilon)},$$

where $\phi(x, \epsilon) = \det(1 - \epsilon AX)$, with $X = \text{diag}(x_1, \dots, x_N)$, is an even polynomial in ϵ .

Proof. Equations (2) can be put as

$$\tilde{x} = A^{-1}(x, \epsilon)(x + \epsilon c), \quad A(x, \epsilon) = 1 - \epsilon AX.$$

As for any Kahan type discretization, there holds the formula

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\det A(\tilde{x}, -\epsilon)}{\det A(x, \epsilon)}. \quad (3)$$

Indeed, differentiation of equations (2) with respect to x_j will give the j -th column of the matrix equation

$$A(x, \epsilon) \frac{\partial \tilde{x}}{\partial x} = A(\tilde{x}, -\epsilon).$$

It remains to notice that $\det A(x, \epsilon) = \det A(x, -\epsilon)$. Indeed, due to the skew-symmetry of A , we have: $\det(1 - \epsilon AX) = \det(1 - \epsilon X^T A^T) = \det(1 + \epsilon XA) = \det(1 + \epsilon AX)$. \square

The second class consists of equations of the Lotka-Volterra type:

$$\dot{x}_i = x_i \left(b_i + \sum_{j=1}^N a_{ij} x_j \right), \quad 1 \leq i \leq N,$$

with a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^N = -A^T$. The Kahan's discretization (with the stepsize 2ϵ) reads:

$$\tilde{x}_i - x_i = \epsilon b_i (x_i + \tilde{x}_i) + \epsilon \sum_{j=1}^N a_{ij} (x_i \tilde{x}_j + \tilde{x}_i x_j), \quad 1 \leq i \leq N. \quad (4)$$

Proposition 2. *The map $\tilde{x} = f(x, \epsilon)$ defined by equations (4) has an invariant volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\tilde{x}_1 \tilde{x}_2 \cdots \tilde{x}_N}{x_1 x_2 \cdots x_N} \Leftrightarrow f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge \cdots \wedge dx_N}{x_1 x_2 \cdots x_N}.$$

Proof. Equations (4) are equivalent to

$$\frac{\tilde{x}_i}{1 + \epsilon b_i + \epsilon \sum_{j=1}^N a_{ij} \tilde{x}_j} = \frac{x_i}{1 - \epsilon b_i - \epsilon \sum_{j=1}^N a_{ij} x_j} =: y_i. \quad (5)$$

We denote $d_i(x, \epsilon) = 1 - \epsilon b_i - \epsilon \sum_{j=1}^N a_{ij} x_j$. In the matrix form equation (4) can be put as

$$\tilde{x} = A^{-1}(x, \epsilon)(\mathbb{1} + \epsilon B)x,$$

where the i -th diagonal entry of $A(x, \epsilon)$ equals $d_i(x, \epsilon)$, while the (i, j) -th off-diagonal entry equals $-\epsilon x_i a_{ij}$. In other words, $A(x, \epsilon) = D(\mathbb{1} - \epsilon Y A)$, where $D = D(x, \epsilon) = \text{diag}(d_1, \dots, d_N)$ and $Y = \text{diag}(y_1, \dots, y_N)$. Formula (3) holds true also in the present case, and it implies:

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\det D(\tilde{x}, -\epsilon)}{\det D(x, \epsilon)} \frac{\det(\mathbb{1} + \epsilon Y A)}{\det(\mathbb{1} - \epsilon Y A)}.$$

The second factor equals 1 due to the skew-symmetry of A , while the first factor equals

$$\frac{d_1(\tilde{x}, -\epsilon) \cdots d_N(\tilde{x}, -\epsilon)}{d_1(x, \epsilon) \cdots d_N(x, \epsilon)} = \frac{\tilde{x}_1 \cdots \tilde{x}_N}{x_1 \cdots x_N},$$

by virtue of (5). □

The statement of Theorem 2 for the Kahan discretization of the classical Lotka-Volterra model with two species, $\dot{x} = x(a - by)$, $\dot{y} = y(cx - d)$, was found in [23] and used to explain the non-spiralling behavior of the numerical orbits in this case.

3. HIROTA-KIMURA BASES AND INTEGRALS

In this section a general formulation of a remarkable mechanism will be given, which seems to be responsible for the integrability of the Hirota-Kimura type (or Kahan type) discretizations of algebraically completely integrable systems. This mechanism is so far not well understood, in fact at the moment we do not know what mathematical structures make it actually work.

Throughout this section $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a birational map, while $h_i, \varphi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ stand for rational, usually polynomial functions on the phase space. We start with recalling a well known definition.

Definition 3. *A function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is called an **integral**, or a **conserved quantity**, of the map f , if for every $x \in \mathbb{R}^n$ there holds*

$$h(f(x)) = h(x),$$

so that $h(f^i(x)) = h(x)$ for all $i \in \mathbb{Z}$.

Thus, each orbit of the map f lies on a certain level set of its integral h . As a consequence, if one knows d functionally independent integrals h_1, \dots, h_d of f , one can claim that each orbit of f is confined to an $(n - d)$ -dimensional invariant set, which is a common level set of the functions h_1, \dots, h_d .

Definition 4. A set of functions $\Phi = (\varphi_1, \dots, \varphi_l)$, linearly independent over \mathbb{R} , is called a **Hirota-Kimura basis (HK basis)**, if for every $x \in \mathbb{R}^n$ there exists a vector $c = (c_1, \dots, c_l) \neq 0$ such that

$$c_1\varphi_1(f^i(x)) + \dots + c_l\varphi_l(f^i(x)) = 0 \quad (6)$$

holds true for all $i \in \mathbb{Z}$. For a given $x \in \mathbb{R}^n$, the vector space consisting of all $c \in \mathbb{R}^l$ with this property will be denoted by $K_\Phi(x)$ and called the null-space of the basis Φ (at the point x).

Thus, for a HK basis Φ and for $c \in K_\Phi(x)$ the function $h = c_1\varphi_1 + \dots + c_l\varphi_l$ vanishes along the f -orbit of x . Let us stress that we cannot claim that $h = c_1\varphi_1 + \dots + c_l\varphi_l$ is an integral of motion, since vectors $c \in K_\Phi(x)$ do not have to belong to $K_\Phi(y)$ for initial points y not lying on the orbit of x . However, for any x the orbit $\{f^i(x)\}$ is confined to the common zero level set of d functions

$$h_j = c_1^{(j)}\varphi_1 + \dots + c_l^{(j)}\varphi_l = 0, \quad j = 1, \dots, d,$$

where the vectors $c^{(j)} = (c_1^{(j)}, \dots, c_l^{(j)}) \in \mathbb{R}^l$ form a basis of $K_\Phi(x)$. We will say that the HK basis Φ is **regular**, if the differentials dh_1, \dots, dh_d are linearly independent along the common zero level set of the functions h_1, \dots, h_d . Thus, knowledge of a regular HK basis with a d -dimensional null-space leads to a similar conclusion as knowledge of d independent integrals of f , namely to the conclusion that the orbits lie on $(n-d)$ -dimensional invariant sets. Note, however, that a HK basis gives no immediate information on how these invariant sets foliate the phase space \mathbb{R}^n , since the vectors $c^{(j)}$, and therefore the functions h_j , change from one initial point x to another.

Although the notions of integrals and of HK bases cannot be immediately translated into one another, they turn out to be closely related.

The simplest situation for a HK basis corresponds to $l = 2$, $\dim K_\Phi(x) = d = 1$. In this case we immediately see that $h = \varphi_1/\varphi_2$ is an integral of motion of the map f . Conversely, for any rational integral of motion $h = \varphi_1/\varphi_2$ its numerator and denominator φ_1, φ_2 satisfy

$$c_1\varphi_1(f^i(x)) + c_2\varphi_2(f^i(x)) = 0, \quad i \in \mathbb{Z},$$

with $c_1 = 1, c_2 = -h(x)$, and thus build a HK basis with $l = 2$. Thus, the notion of a HK basis *generalizes* (for $l \geq 3$) the notion of integrals of motion.

On the other hand, knowing a HK basis Φ with $\dim K_\Phi(x) = d \geq 1$ allows one to find integrals of motion for the map f . Indeed, from Definition 4 there follows immediately:

Proposition 5. *If Φ is a HK basis for a map f , then*

$$K_\Phi(f(x)) = K_\Phi(x).$$

Thus, the d -dimensional null-space $K_\Phi(x) \in Gr(d, l)$, regarded as a function of the initial point $x \in \mathbb{R}^n$, is constant along trajectories of the map f , i.e., it is a $Gr(d, l)$ -valued integral. Its Plücker coordinates are then scalar integrals:

Corollary 6. *Let Φ be a HK basis for f with $\dim K_\Phi(x) = d$ for all $x \in \mathbb{R}^n$. Take a basis of $K_\Phi(x)$ consisting of d vectors $c^{(i)} \in \mathbb{R}^l$ and put them into the columns of a $l \times d$ matrix $C(x)$. For any d -index $\alpha = (\alpha_1, \dots, \alpha_d) \subset \{1, 2, \dots, n\}$ let $C_\alpha = C_{\alpha_1 \dots \alpha_d}$ denote the $d \times d$ minor of the matrix C built from the rows $\alpha_1, \dots, \alpha_d$. Then for any two d -indices α, β the function C_α/C_β is an integral of f .*

Especially simple is the situation when the null-space of a HK basis has dimension $d = 1$.

Corollary 7. *Let Φ be a HK basis for f with $\dim K_\Phi(x) = 1$ for all $x \in \mathbb{R}^n$. Let $K_\Phi(x) = [c_1(x) : \dots : c_l(x)] \in \mathbb{RP}^{l-1}$. Then the functions c_j/c_k are integrals of motion for f .*

An interesting (and difficult) question is about the number of functionally independent integrals obtained from a given HK basis according to Corollaries 6 and 7. It is possible for a HK basis with a one-dimensional null-space to produce more than one independent integral. The first examples of this mechanism (with $d = 1$) were found in [12] and (somewhat implicitly) in [8].

We note, however, that HK bases appeared in a disguised form in the continuous time theory long ago. We mention here two relevant examples.

- Classically, integration of a given system of ODEs in terms of elliptic functions started with the derivation of an equation of the type $\dot{y}^2 = P_4(y)$, where y is one of the components of the solution, and $P_4(y)$ is a polynomial of degree 4 with constant coefficients (depending on parameters of the system and on its integrals of motion), see examples in Sections 12, 13. This can be interpreted as the claim about $\Phi = (\dot{y}^2, y^4, y^3, y^2, y, 1)$ being a HK basis with a one-dimensional null-space.
- According to [1, Sect. 7.6.6], for any algebraically integrable system, one can choose projective coordinates y_0, y_1, \dots, y_n so that *quadratic Wronskian equations* are satisfied:

$$\dot{y}_i y_j - y_i \dot{y}_j = \sum_{k,l=0}^n \alpha_{ij}^{kl} y_k y_l,$$

with coefficients α_{ij}^{kl} depending on integrals of motion of the original system. Again, this admits an immediate interpretation in terms of HK bases consisting of the Wronskians and the quadratic monomials of the coordinate functions: $\Phi_{ij} = (\dot{y}_i y_j - y_i \dot{y}_j, \{y_k y_l\}_{k,l=0}^n)$.

Thus, these HK bases consist not only of simple monomials, but include also more complicated functions composed of the vector field of the system at hand. We will encounter discrete counterparts of these HK bases, as well.

At present, we cannot give any theoretical sufficient conditions for existence of a HK basis Φ for a given map f , and the only way to find such a basis remains the experimental one. Definition 4 requires to verify condition (6) for all $i \in \mathbb{Z}$, which is, of course, impractical. However, it is enough to check this condition for a finite number of iterates f^i .

Typically (that is, for general maps f and general monomial sets Φ), the dimension of the vector space of solutions of the homogeneous system of s linear equations (6) with $i = i_0, i_0 + 1, \dots, i_0 + s - 1$ decays with the growing s , from $l - 1$ for $s = 1$ down to 0 for $s = l$. If, however, Φ is a HK basis for f with a d -dimensional null-space, then this dimension fails to drop starting with $s = l - d + 1$. Thus, the dimension of the solution space of the system (6) with $i = i_0, i_0 + 1, \dots, i_0 + s - 1$ is equal to $l - s$ for all $1 \leq s \leq l - d$, and remains equal to d for $s = l - d + 1$. It is easy to see that this situation can be also characterized as follows: the d -dimensional spaces of solutions of the system (6) with $i = i_0, i_0 + 1, \dots, i_0 + l - d - 1$ and of the system (6) with $i = i_0 + 1, i_0 + 2, \dots, i_0 + l - d$

coincide with each other and with $K_\Phi(x)$. The most important particular case of this characterization corresponds to $d = 1$ and reads as follows:

Proposition 8. *A set $\Phi = (\varphi_1, \dots, \varphi_l)$ is a HK basis for a map f with $\dim K_\Phi(x) = 1$ if and only if the unique solution $[c_1 : \dots : c_l] \in \mathbb{RP}^{l-1}$ of the system (6) with $i = i_0, i_0 + 1, \dots, i_0 + l - 2$ coincides with the unique solution of the analogous system with $i = i_0 + 1, i_0 + 2, \dots, i_0 + l - 1$, in other words, if this unique solution $[c_1 : \dots : c_l]$ is an integral of motion:*

$$[c_1(x) : \dots : c_l(x)] = [c_1(f(x)) : \dots : c_l(f(x))].$$

At this point it should be mentioned that a numerical testing of the above criterium usually represents no problems, but the corresponding symbolic computations might be extremely complicated, due to the complexity of the iterates $f^i(x)$. While the expression for $f(x)$ is typically of a moderate size, already the second iterate $f^2(x)$ becomes typically prohibitively big. See the detailed discussion of the complexity issue for the case of the Clebsch system in [17]. In such a situation it becomes crucial to reduce the number of iterates involved in (6) as far as possible. Several tricks which can be used for this aim are also described in [17]. Here is one of them.

Proposition 9. *Consider the non-homogeneous system of $l - 1$ equations*

$$c_1\varphi_1(f^i(x)) + \dots + c_{l-1}\varphi_{l-1}(f^i(x)) = \varphi_l(f^i(x)), \quad i = i_0, i_0 + 1, \dots, i_0 + l - 2. \quad (7)$$

Suppose that the index range $i \in [i_0, i_0 + l - 2]$ in eq. (7) contains 0 but is non-symmetric. If the solution of this system $(c_1(x, \epsilon), \dots, c_{l-1}(x, \epsilon))$ is unique and is even with respect to ϵ , then all $c_k(x, \epsilon)$ are conserved quantities of the map f , and $\Phi = (\varphi_1, \dots, \varphi_l)$ is a HK basis for f with $\dim K_\Phi(x) = 1$.

Proof. Considering the non-homogeneous system (7) instead of the homogeneous one (6) corresponds just to fixing an affine representative of the projective solution $[c_1 : \dots : c_l]$ by $c_l = -1$. The reversibility of the map $f^{-1}(x, \epsilon) = f(x, -\epsilon)$ yields that equations of the system (7) are satisfied not only for $i \in [i_0, i_0 + l - 2]$ but for $i \in [-(i_0 + l - 2), -i_0]$, as well. Since, by condition, the intervals $[i_0, i_0 + l - 2]$ and $[-(i_0 + l - 2), -i_0]$ overlap but do not coincide, their union is an interval containing $\geq l$ integers. \square

Of course, it would be highly desirable to find some structures, like Lax representation, bi-Hamiltonian structure, etc., which would allow one to check the conservation of integrals in a more clever way, but up to now no such structures have been found for any of the HK type discretizations.

4. WEIERSTRASS DIFFERENTIAL EQUATION

Consider the second-order differential equation

$$\ddot{x} = 6x^2 - \alpha. \quad (8)$$

Its general solution is given by the Weierstrass elliptic function $\wp(t) = \wp(t, g_2, g_3)$ with the invariants $g_2 = 2\alpha$, g_3 arbitrary, and by its time shifts. Actually, the parameter g_3 can be interpreted as the value of an integral of motion (conserved quantity) of system (8):

$$\dot{x}^2 - 4x^3 + 2\alpha x = -g_3.$$

Being re-written as a system of first-order equations with a quadratic vector field,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 6x^2 - \alpha, \end{cases} \quad (9)$$

equation (8) becomes suitable for an application of the Kahan-Hirota-Kimura discretization:

$$\begin{cases} \tilde{x} - x = \frac{\epsilon}{2} (\tilde{y} + y), \\ \tilde{y} - y = \epsilon (6x\tilde{x} - \alpha). \end{cases} \quad (10)$$

Eqs. (10), put as a linear system for (\tilde{x}, \tilde{y}) , read:

$$\begin{pmatrix} 1 & -\epsilon/2 \\ -6\epsilon x & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + \epsilon y/2 \\ y - \epsilon \alpha \end{pmatrix}.$$

This can be immediately solved, thus yielding an explicit birational map $(\tilde{x}, \tilde{y}) = f(x, y, \epsilon)$:

$$\begin{cases} \tilde{x} = \frac{x + \epsilon y - \epsilon^2 \alpha/2}{1 - 3\epsilon^2 x}, \\ \tilde{y} = \frac{y + \epsilon(6x^2 - \alpha) + 3\epsilon^2 xy}{1 - 3\epsilon^2 x}. \end{cases} \quad (11)$$

This map turns out to be integrable: it possesses an invariant two-form

$$\omega = \frac{dx \wedge dy}{1 - 3\epsilon^2 x}, \quad (12)$$

and an integral of motion (conserved quantity):

$$I(x, y, \epsilon) = \frac{y^2 - 4x^3 + 2\alpha x + \epsilon^2 x(y^2 - 2\alpha x) - \epsilon^4 \alpha^2 x}{1 - 3\epsilon^2 x}. \quad (13)$$

Both these objects are $O(\epsilon^2)$ -perturbations of the corresponding objects for the continuous time system (9). The statement about the invariant 2-form (12) is not difficult to prove. The following argument exemplifies considerations which hold for an arbitrary Kahan discretization (1). Differentiating eqs. (10) with respect to x and to y , we obtain the columns of the matrix equation

$$\begin{pmatrix} 1 & -\epsilon/2 \\ -6\epsilon x & 1 \end{pmatrix} \frac{\partial(\tilde{x}, \tilde{y})}{\partial(x, y)} = \begin{pmatrix} 1 & \epsilon/2 \\ 6\epsilon \tilde{x} & 1 \end{pmatrix},$$

whence

$$\det \frac{\partial(\tilde{x}, \tilde{y})}{\partial(x, y)} = \frac{1 - 3\epsilon^2 \tilde{x}}{1 - 3\epsilon^2 x}.$$

This is equivalent to the preservation of (12). The statement about the conserved quantity is most simply verified with any computer system for symbolic manipulations.

System (10) is known in the literature on integrable maps, although in a somewhat different form. Indeed, it is equivalent to the second order difference equation

$$\tilde{x} - 2x + \underline{x} = \epsilon^2 (3x(\tilde{x} + \underline{x}) - \alpha) \quad \Leftrightarrow \quad \tilde{x} - 2x + \underline{x} = \frac{\epsilon^2 (6x^2 - \alpha)}{1 - 3\epsilon^2 x}.$$

This equation belongs to class of integrable *QRT systems* [20, 24]; in order to see this, one should re-write it as

$$\tilde{x} - 2x + \underline{x} = \frac{\epsilon^2 (6x^2 - \alpha)(1 + \epsilon^2 x)}{1 - 2\epsilon^2 x - 3\epsilon^4 x^2}.$$

This difference equation generates a map $(x, \underline{x}) \mapsto (\tilde{x}, x)$ which is symplectic, that is, preserves the two-form $\omega = dx \wedge d\tilde{x}$, and possesses a biquadratic integral of motion

$$I(x, \tilde{x}, \epsilon) = (\tilde{x} - x)^2 - 2\epsilon^2 x \tilde{x}(x + \tilde{x}) + \epsilon^2 \alpha(x + \tilde{x}) - \epsilon^4(3x^2 \tilde{x}^2 - \alpha x \tilde{x}).$$

Under the change of variables $(x, \tilde{x}) \mapsto (x, y)$ given by the first equation in (11), these integrability attributes turn into the two-form (12) and the conserved quantity (13) (up to an additive constant).

We note that a more usual QRT discretization of the Weierstrass second order equation (8) would be

$$\tilde{x} - 2x + \underline{x} = \frac{\epsilon^2(6x^2 - \alpha)}{1 - 2\epsilon^2 x}, \quad (14)$$

with a simpler conserved quantity

$$J(x, \tilde{x}, \epsilon) = (\tilde{x} - x)^2 - 2\epsilon^2 x \tilde{x}(x + \tilde{x}) + \epsilon^2 \alpha(x + \tilde{x}).$$

Eq. (14) is equivalent to

$$\tilde{x} - 2x + \underline{x} = \epsilon^2(2x(\tilde{x} + \underline{x}) + 2x^2 - \alpha),$$

which is not obtained by the Kahan-Hirota-Kimura method.

5. SOME TWO-DIMENSIONAL INTEGRABLE SYSTEMS

5.1. The three-dimensional Suslov system. The three-dimensional nonholonomic Suslov problem [26] is defined by the following system of differential equations

$$\dot{m} = m \times \omega + \lambda a, \quad \langle a, \omega \rangle = 0, \quad (15)$$

where $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the angular velocity, $m = (m_1, m_2, m_3)^T = I\omega$ is the angular momentum, I is the inertia operator, a is a unit vector fixed in the body, and λ is the Lagrange multiplier. In a basis where $a = (0, 0, 1)^T$ and

$$I = \begin{pmatrix} I_1 & 0 & I_{13} \\ 0 & I_2 & I_{23} \\ I_{13} & I_{23} & I_3 \end{pmatrix},$$

the constraint $\langle a, \omega \rangle = 0$ reduces to $\omega_3 = 0$, and equations of motion (15) read

$$\begin{cases} I_1 \dot{\omega}_1 = -(I_{13}\omega_1 + I_{23}\omega_2)\omega_2, \\ I_2 \dot{\omega}_2 = (I_{13}\omega_1 + I_{23}\omega_2)\omega_1, \\ I_{13}\dot{\omega}_1 + I_{23}\dot{\omega}_2 = (I_1 - I_2)\omega_1\omega_2 + \lambda. \end{cases} \quad (16)$$

The first two equations in (16) form a closed system for ω_1 and ω_2 . It possesses a conserved quantity $H = I_1\omega_1^2 + I_2\omega_2^2$. After the solution of this system is found (Suslov gave it in terms of trigonometric and exponential functions), one finds the Lagrange multiplier λ from the third equation in (16).

To put the first two equations in (16) into a more convenient form, one can introduce the coordinates $x = I_{13}\omega_1 + I_{23}\omega_2$, $y = I_{23}I_1\omega_1 - I_{13}I_2\omega_2$, and arrives at

$$\begin{cases} \dot{x} = \alpha xy, \\ \dot{y} = -x^2, \end{cases} \quad (17)$$

where $\alpha = 1/I_1I_2$. This system admits a conserved quantity

$$H = x^2 + \alpha y^2.$$

Proposition 10. [5] *HK discretization of system (17),*

$$\begin{cases} \tilde{x} - x = \epsilon\alpha(\tilde{x}y + x\tilde{y}), \\ \tilde{y} - y = -2\epsilon\tilde{x}x, \end{cases}$$

possesses an invariant two-form and an integral of motion, given by

$$\omega = \frac{dx \wedge dy}{x(x^2 + \alpha y^2)}, \quad H(\epsilon) = \frac{x^2 + \alpha y^2}{1 + \epsilon^2 \alpha x^2}.$$

Actually, the invariant two-form was not given in [5]; rather, this paper contains an explicit solution of the discrete time Suslov system and its qualitative analysis.

5.2. Reduced Nahm equations. In [9] Nahm equations associated with symmetric monopoles are considered. Assuming rotational symmetry groups of regular polytopes leads to solutions of Nahm equations in terms of elliptic functions. Reduced equations corresponding to tetrahedrally symmetric monopoles of charge 3, to octahedrally symmetric monopoles of charge 4, and to icosahedrally symmetric monopoles of charge 6 are two-dimensional algebraically integrable systems with quadratic vector fields. More concretely, the reductions of Nahm equations derived in [9] read:

(i) tetrahedral symmetry:

$$\begin{cases} \dot{x} = x^2 - y^2, \\ \dot{y} = -2xy, \end{cases} \quad (18)$$

with an integral of motion $H = y(3x^2 - y^2)$;

(ii) octahedral symmetry:

$$\begin{cases} \dot{x} = 2x^2 - 12y^2, \\ \dot{y} = -6xy - 4y^2, \end{cases} \quad (19)$$

with an integral of motion $H = y(2x + 3y)(x - y)^2$;

(iii) icosahedral symmetry:

$$\begin{cases} \dot{x} = 2x^2 - y^2, \\ \dot{y} = -10xy + y^2, \end{cases} \quad (20)$$

with an integral of motion $H = y(3x - y)^2(4x + y)^3$.

HK discretizations of systems (18-20) turn out to be algebraically integrable.

Proposition 11.

(i) *HK discretization of system (18),*

$$\begin{cases} \tilde{x} - x = \epsilon(\tilde{x}x - \tilde{y}y), \\ \tilde{y} - y = -\epsilon(\tilde{x}y + x\tilde{y}), \end{cases}$$

possesses an invariant two-form and an integral of motion, given by

$$\omega = \frac{dx \wedge dy}{y(3x^2 - y^2)}, \quad H(\epsilon) = \frac{y(3x^2 - y^2)}{1 - \epsilon^2(x^2 + y^2)};$$

(ii) *HK discretization of system (19),*

$$\begin{cases} \tilde{x} - x = \epsilon(2\tilde{x}x - 12\tilde{y}y), \\ \tilde{y} - y = -\epsilon(3\tilde{x}y + 3x\tilde{y} + 4\tilde{y}y), \end{cases}$$

possesses an invariant two-form and an integral of motion given by

$$\omega = \frac{dx \wedge dy}{y(2x + 3y)(x - y)},$$

$$H(\epsilon) = \frac{y(2x + 3y)(x - y)^2}{1 - 10\epsilon^2(x^2 + 4y^2) + \epsilon^4(9x^4 + 272x^3y - 352xy^3 + 696y^4)};$$

(iii) *HK discretization of system (20),*

$$\begin{cases} \tilde{x} - x = \epsilon(2\tilde{x}x - \tilde{y}y), \\ \tilde{y} - y = \epsilon(-5\tilde{x}y - 5x\tilde{y} + \tilde{y}y), \end{cases}$$

possesses an invariant two-form and an integral of motion given by

$$\omega = \frac{dx \wedge dy}{y(3x - y)(4x + y)}, \quad H(\epsilon) = \frac{y(3x - y)^2(4x + y)^3}{1 + \epsilon^2c_2 + \epsilon^4c_4 + \epsilon^6c_6},$$

with

$$\begin{aligned} c_2 &= -7(5x^2 - y^2), \\ c_4 &= 7(37x^4 + 22x^2y^2 - 2xy^3 + 2y^4), \\ c_6 &= -225x^6 + 3840x^5y + 80xy^5 - 514x^3y^3 - 19x^4y^2 - 206x^2y^4. \end{aligned}$$

6. EULER TOP

The differential equations of motion of the Euler top read

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{cases} \quad (21)$$

with real parameters α_i . This is one of the most famous integrable systems of the classical mechanics, with a big literature devoted to it. It can be explicitly integrated in terms of elliptic functions, and admits two functionally independent integrals of motion. Actually, a quadratic function $H(x) = \gamma_1 x_1^2 + \gamma_2 x_2^2 + \gamma_3 x_3^2$ is an integral for eqs. (21) as soon as $\gamma_1 \alpha_1 + \gamma_2 \alpha_2 + \gamma_3 \alpha_3 = 0$. In particular, the following three functions are integrals of motion:

$$H_1 = \alpha_2 x_3^2 - \alpha_3 x_2^2, \quad H_2 = \alpha_3 x_1^2 - \alpha_1 x_3^2, \quad H_3 = \alpha_1 x_2^2 - \alpha_2 x_1^2.$$

Clearly, only two of them are functionally independent because of $\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3 = 0$. These integrals appear also on the right-hand sides of the quadratic (in this case even linear) expressions for the Wronskians of the coordinates x_j :

$$\begin{cases} \dot{x}_2 x_3 - x_2 \dot{x}_3 = H_1 x_1, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 = H_2 x_2, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3. \end{cases} \quad (22)$$

Moreover, one easily sees that the coordinates x_j satisfy the following differential equations with the coefficients depending on the integrals of motion:

$$\begin{cases} \dot{x}_1^2 = (H_3 + \alpha_2 x_1^2)(\alpha_3 x_1^2 - H_2), \\ \dot{x}_2^2 = (H_1 + \alpha_3 x_2^2)(\alpha_1 x_2^2 - H_3), \\ \dot{x}_3^2 = (H_2 + \alpha_1 x_3^2)(\alpha_2 x_3^2 - H_1). \end{cases}$$

The fact that the polynomials on the right-hand sides of these equations are of degree four implies that the solutions are given by elliptic functions.

The HK discretization of the Euler top [8] is:

$$\begin{cases} \tilde{x}_1 - x_1 = \epsilon \alpha_1 (\tilde{x}_2 x_3 + x_2 \tilde{x}_3), \\ \tilde{x}_2 - x_2 = \epsilon \alpha_2 (\tilde{x}_3 x_1 + x_3 \tilde{x}_1), \\ \tilde{x}_3 - x_3 = \epsilon \alpha_3 (\tilde{x}_1 x_2 + x_1 \tilde{x}_2). \end{cases} \quad (23)$$

(In this form it corresponds to the stepsize 2ϵ rather than ϵ .) The map $f : x \mapsto \tilde{x}$ obtained by solving (23) for \tilde{x} is given by:

$$\tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x, \quad A(x, \epsilon) = \begin{pmatrix} 1 & -\epsilon \alpha_1 x_3 & -\epsilon \alpha_1 x_2 \\ -\epsilon \alpha_2 x_3 & 1 & -\epsilon \alpha_2 x_1 \\ -\epsilon \alpha_3 x_2 & -\epsilon \alpha_3 x_1 & 1 \end{pmatrix}. \quad (24)$$

It might be instructive to have a look at the explicit formulas for this map:

$$\begin{cases} \tilde{x}_1 = \frac{x_1 + 2\epsilon \alpha_1 x_2 x_3 + \epsilon^2 x_1 (-\alpha_2 \alpha_3 x_1^2 + \alpha_3 \alpha_1 x_2^2 + \alpha_1 \alpha_2 x_3^2)}{\Delta(x, \epsilon)}, \\ \tilde{x}_2 = \frac{x_2 + 2\epsilon \alpha_2 x_3 x_1 + \epsilon^2 x_2 (\alpha_2 \alpha_3 x_1^2 - \alpha_3 \alpha_1 x_2^2 + \alpha_1 \alpha_2 x_3^2)}{\Delta(x, \epsilon)}, \\ \tilde{x}_3 = \frac{x_3 + 2\epsilon \alpha_3 x_1 x_2 + \epsilon^2 x_3 (\alpha_2 \alpha_3 x_1^2 + \alpha_3 \alpha_1 x_2^2 - \alpha_1 \alpha_2 x_3^2)}{\Delta(x, \epsilon)}, \end{cases} \quad (25)$$

where

$$\Delta(x, \epsilon) = \det A(x, \epsilon) = 1 - \epsilon^2 (\alpha_2 \alpha_3 x_1^2 + \alpha_3 \alpha_1 x_2^2 + \alpha_1 \alpha_2 x_3^2) - 2\epsilon^3 \alpha_1 \alpha_2 \alpha_3 x_1 x_2 x_3. \quad (26)$$

We will use the abbreviation dET for this map. As always the case for a HK discretization, dET is birational, with the reversibility property expressed as $f^{-1}(x, \epsilon) = f(x, -\epsilon)$.

Proposition 12. [8, 18] *The quantities*

$$F_1 = \frac{1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2}{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2}, \quad F_2 = \frac{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2}{1 - \epsilon^2 \alpha_2 \alpha_3 x_1^2}, \quad F_3 = \frac{1 - \epsilon^2 \alpha_2 \alpha_3 x_1^2}{1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2},$$

are conserved quantities of dET. Of course, there are only two independent integrals since $F_1 F_2 F_3 = 1$.

The relation between F_i and the integrals H_i of the continuous time Euler top is straightforward: $F_i = 1 + \epsilon^2 \alpha_i H_i + O(\epsilon^4)$. As a corollary of Proposition 12, we find that, for any conserved quantity H of the Euler top which is a linear combination of the integrals H_1, H_2, H_3 , the three functions $H/(1 - \epsilon^2 \alpha_j \alpha_k x_i^2)$ are conserved quantities of dET. Hereafter (i, j, k) are cyclic permutations of $(1, 2, 3)$. In particular, the functions

$$H_i(\epsilon) = \frac{\alpha_j x_k^2 - \alpha_k x_j^2}{1 - \epsilon^2 \alpha_j \alpha_k x_i^2} \quad (27)$$

are conserved quantities of dET. Again, only two of them are independent, since

$$\alpha_1 H_1(\epsilon) + \alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon) + \epsilon^4 \alpha_1 \alpha_2 \alpha_3 H_1(\epsilon) H_2(\epsilon) H_3(\epsilon) = 0.$$

Proposition 13. [18] *The map dET possesses an invariant volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\phi(\tilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

where $\phi(x)$ is any of the functions

$$\phi(x) = (1 - \epsilon^2 \alpha_i \alpha_j x_k^2)(1 - \epsilon^2 \alpha_j \alpha_k x_i^2) \quad \text{or} \quad (1 - \epsilon^2 \alpha_i \alpha_j x_k^2)^2.$$

(The ratio of any two functions $\phi(x)$ is an integral of motion, due to Proposition 12).

The proof is based on formula (3) with the matrix $A(x, \epsilon)$ given in (24). Its determinant is given in (26).

A proper discretization of the Wronskian differential equations (22) is given by the following statement.

Proposition 14. *The following relations hold true for dET:*

$$\begin{cases} \tilde{x}_2 x_3 - x_2 \tilde{x}_3 = \epsilon H_1(\epsilon)(\tilde{x}_1 + x_1), \\ \tilde{x}_3 x_1 - x_3 \tilde{x}_1 = \epsilon H_2(\epsilon)(\tilde{x}_3 + x_3), \\ \tilde{x}_1 x_2 - x_1 \tilde{x}_2 = \epsilon H_3(\epsilon)(\tilde{x}_3 + x_3), \end{cases} \quad (28)$$

with the functions $H_i(\epsilon)$ from (27).

The proof is based on relations

$$\tilde{x}_i + x_i = \frac{2(1 - \epsilon^2 \alpha_j \alpha_k x_i^2)(x_i + \epsilon \alpha_i x_j x_k)}{\Delta(x, \epsilon)}, \quad (29)$$

$$\tilde{x}_j x_k - x_j \tilde{x}_k = \frac{2\epsilon(\alpha_j x_k^2 - \alpha_k x_j^2)(x_i + \epsilon \alpha_i x_j x_k)}{\Delta(x, \epsilon)}, \quad (30)$$

which follow easily from the explicit formulas (25). They should be compared with

$$\tilde{x}_i - x_i = \epsilon \alpha_i (\tilde{x}_j x_k + x_j \tilde{x}_k) = \frac{2\epsilon \alpha_i (x_j + \epsilon \alpha_j x_k x_i)(x_k + \epsilon \alpha_k x_i x_j)}{\Delta(x, \epsilon)}. \quad (31)$$

As pointed out in [17], a probable way to the discovery of the conserved quantities of dET in [8] was through finding the HK bases for this map. In this respect, one has the following results.

Proposition 15. [17]

(a) *The set $\Phi = (x_1^2, x_2^2, x_3^2, 1)$ is a HK basis for dET with $\dim K_\Phi(x) = 2$. Therefore, any orbit of dET lies on the intersection of two quadrics in \mathbb{R}^3 .*

(b) *The set $\Phi_0 = (x_1^2, x_2^2, x_3^2)$ is a HK basis for dET with $\dim K_{\Phi_0}(x) = 1$. At each point $x \in \mathbb{R}^3$ we have:*

$$K_{\Phi_0}(x) = [c_1 : c_2 : c_3] = [\alpha_2 x_3^2 - \alpha_3 x_2^2 : \alpha_3 x_1^2 - \alpha_1 x_3^2 : \alpha_1 x_2^2 - \alpha_2 x_1^2].$$

Setting $c_3 = -1$, the following functions are integrals of motion of dET:

$$c_1(x) = \frac{\alpha_3 x_2^2 - \alpha_2 x_3^2}{\alpha_1 x_2^2 - \alpha_2 x_1^2}, \quad c_2(x) = \frac{\alpha_1 x_3^2 - \alpha_3 x_1^2}{\alpha_1 x_2^2 - \alpha_2 x_1^2}. \quad (32)$$

(c) *The set $\Phi_{12} = (x_1^2, x_2^2, 1)$ is a further HK basis for dET with $\dim K_{\Phi_{12}}(x) = 1$. At each point $x \in \mathbb{R}^3$ we have: $K_{\Phi_{12}}(x) = [d_1 : d_2 : -1]$, where*

$$d_1(x) = -\frac{\alpha_2(1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}, \quad d_2(x) = \frac{\alpha_1(1 - \epsilon^2 \alpha_2 \alpha_3 x_1^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}. \quad (33)$$

These functions are integrals of motion of dET independent on the integrals (32). We have: $K_\Phi(x) = K_{\Phi_0} \oplus K_{\Phi_{12}}$.

Proof. To prove statement (b), we solve the system

$$\begin{cases} c_1 x_1^2 + c_2 x_2^2 = x_3^2, \\ c_1 \tilde{x}_1^2 + c_2 \tilde{x}_2^2 = \tilde{x}_3^2. \end{cases}$$

The solution is given, according to the Cramer's rule, by ratios of determinants of the type

$$\begin{vmatrix} x_i^2 & x_j^2 \\ \tilde{x}_i^2 & \tilde{x}_j^2 \end{vmatrix} = \frac{4\epsilon(\alpha_j x_i^2 - \alpha_i x_j^2)(x_1 + \epsilon\alpha_1 x_2 x_3)(x_2 + \epsilon\alpha_2 x_3 x_1)(x_3 + \epsilon\alpha_3 x_1 x_2)}{\Delta^2(x, \epsilon)}. \quad (34)$$

(Here we used (30), (31)). In the ratios of such determinants everything cancels out, except for the factors $\alpha_j x_i^2 - \alpha_i x_j^2$, so we end up with (32). The cancelation of the denominators $\Delta^2(x, \epsilon)$ is, of course, no wonder, but the cancelation of all the non-even factors in the numerators is rather *remarkable and miraculous* and is not granted by any well-understood mechanism. Since the components of the solution do not depend on ϵ , we conclude that functions (32) are integrals of motion of dET.

To prove statement (c), we solve the system

$$\begin{cases} d_1 x_1^2 + d_2 x_2^2 = 1, \\ d_1 \tilde{x}_1^2 + d_2 \tilde{x}_2^2 = 1. \end{cases}$$

The solution is given by eq. (33), due to eq. (34) and the similar formula

$$\begin{vmatrix} 1 & x_i^2 \\ 1 & \tilde{x}_i^2 \end{vmatrix} = \frac{4\epsilon\alpha_i(1 - \epsilon^2\alpha_j\alpha_k x_i^2)(x_1 + \epsilon\alpha_1 x_2 x_3)(x_2 + \epsilon\alpha_2 x_3 x_1)(x_3 + \epsilon\alpha_3 x_1 x_2)}{\Delta^2(x, \epsilon)},$$

which, in turn, follows from (29) and (30). This time the solution does depend on ϵ , but consists of manifestly even functions of ϵ . Everything non-even luckily cancels, again. Therefore, functions (33) are integrals of motion of dET. \square

Although each one of the HK bases Φ_0, Φ_1 delivers apparently two integrals of motion (32), each pair turns out to be *functionally dependent*, as

$$\alpha_1 c_1(x) + \alpha_2 c_2(x) = \alpha_3, \quad \alpha_1 d_1(x) + \alpha_2 d_2(x) = \epsilon^2 \alpha_1 \alpha_2 \alpha_3.$$

However, functions c_1, c_2 are independent on d_1, d_2 , since the former depend on x_3 , while the latter do not.

Of course, permutational symmetry yields that each of the sets of monomials $\Phi_{23} = (x_2^2, x_3^2, 1)$ and $\Phi_{13} = (x_1^2, x_3^2, 1)$ is a HK basis, as well, with $\dim K_{\Phi_{23}}(x) = \dim K_{\Phi_{13}}(x) = 1$. But we do not obtain additional linearly independent null-spaces, as any two of the four found one-dimensional null-spaces span the full null-space $K_\Phi(x)$.

Summarizing, we have found a HK basis with a two-dimensional null-space, as well as two functionally independent conserved quantities for the HK discretization of the Euler top. Both results yield integrability of this discretization, in the sense that its orbits are confined to closed curves in \mathbb{R}^3 . Moreover, each such curve is an intersection of two quadrics, which in the general position case is an elliptic curve.

Proposition 16. *Each component x_i of any solution of dET satisfies a relation of the type $P_i(x_i, \tilde{x}_i) = 0$, where P_i is a biquadratic polynomial whose coefficients are integrals of motion of dET:*

$$P_i(x_i, \tilde{x}_i) = p_i^{(3)} x_i^2 \tilde{x}_i^2 + p_i^{(2)} (x_i^2 + \tilde{x}_i^2) + p_i^{(1)} x_i \tilde{x}_i + p_i^{(0)} = 0,$$

with

$$\begin{aligned} p_i^{(3)} &= -4\epsilon^2 \alpha_j \alpha_k, & p_i^{(2)} &= (1 + \epsilon^2 \alpha_j H_j(\epsilon)) (1 - \epsilon^2 \alpha_k H_k(\epsilon)), \\ p_i^{(1)} &= -2(1 - \epsilon^2 \alpha_j H_j(\epsilon)) (1 + \epsilon^2 \alpha_k H_k(\epsilon)), & p_i^{(0)} &= 4\epsilon^2 H_j(\epsilon) H_k(\epsilon). \end{aligned}$$

Proof. From eqs. (23) and (28) there follows:

$$\frac{(\tilde{x}_i - x_i)^2}{(\epsilon \alpha_i)^2} + \epsilon^2 H_i^2(\epsilon) (\tilde{x}_i + x_i)^2 = 2(\tilde{x}_j^2 x_k^2 + x_j^2 \tilde{x}_k^2).$$

It remains to express x_j^2 and x_k^2 through x_i^2 and integrals $H_j(\epsilon)$, $H_k(\epsilon)$ given in eq. (27). \square

It follows from Proposition 16 that solutions $x_i(t)$ as functions of the discrete time $t \in 2\epsilon\mathbb{Z}$ are given by elliptic functions of order 2 (the order of an elliptic function is the number of the zeroes or poles it possesses in a period parallelogram).

We would like to point out that Propositions 14 and 16 can be interpreted as existence of further HK bases. For instance, according to Proposition 14, each pair $(\tilde{x}_j x_k - x_j \tilde{x}_k, \tilde{x}_i + x_i)$ is a HK basis with a 1-dimensional null-space. Similarly, Proposition 16 says that for each $i = 1, 2, 3$, the set $x_i^p \tilde{x}_i^q$ ($0 \leq p, q \leq 2$) is a HK basis with a 1-dimensional null-space. Of course, due to the dependence on the shifted variables \tilde{x} , these HK bases consist of complicated functions of x rather than of monomials. A further instance of HK bases of this sort is given in the following statement. Compared with Proposition 15, it says that for dET, *for each HK basis consisting of monomials quadratic in x , the corresponding set of monomials bilinear in x, \tilde{x} is a HK basis, as well.* This seems to be a quite general phenomenon, further issues of which will appear later several times.

Proposition 17.

(a) *The set $\Psi = (\tilde{x}_1 x_1, \tilde{x}_2 x_2, \tilde{x}_3 x_3, 1)$ is a HK basis for dET with $\dim K_\Psi(x) = 2$.*

(b) *The set $\Psi_0 = (\tilde{x}_1 x_1, \tilde{x}_2 x_2, \tilde{x}_3 x_3)$ is a HK basis for dET with $\dim K_{\Psi_0}(x) = 1$. At each point $x \in \mathbb{R}^3$, the homogeneous coordinates \bar{c}_i of the null-space $K_{\Psi_0}(x) = [\bar{c}_1 : \bar{c}_2 : \bar{c}_3]$ are given by*

$$\bar{c}_i = (\alpha_j x_k^2 - \alpha_k x_j^2) (1 - \epsilon^2 (\alpha_i \alpha_j x_k^2 + \alpha_k \alpha_i x_j^2 - \alpha_j \alpha_k x_i^2)).$$

The quotients \bar{c}_i/\bar{c}_j are integrals of motion of dET.

(c) *The set $\Psi_{12} = (\tilde{x}_1 x_1, \tilde{x}_2 x_2, 1)$ is a further HK basis for dET with $\dim K_{\Psi_{12}}(x) = 1$. At each point $x \in \mathbb{R}^3$, there holds: $K_{\Psi_{12}}(x) = [\bar{d}_1 : \bar{d}_2 : -1]$, where*

$$\begin{aligned} \bar{d}_1(x) &= -\frac{\alpha_2(1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2} \frac{1 - \epsilon^2 (\alpha_2 \alpha_3 x_1^2 - \alpha_3 \alpha_1 x_2^2 + \alpha_1 \alpha_2 x_3^2)}{1 - \epsilon^2 (\alpha_2 \alpha_3 x_1^2 + \alpha_3 \alpha_1 x_2^2 - \alpha_1 \alpha_2 x_3^2)}, \\ \bar{d}_2(x) &= \frac{\alpha_1(1 - \epsilon^2 \alpha_2 \alpha_3 x_1^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2} \frac{1 - \epsilon^2 (\alpha_3 \alpha_1 x_2^2 - \alpha_2 \alpha_3 x_1^2 + \alpha_1 \alpha_2 x_3^2)}{1 - \epsilon^2 (\alpha_3 \alpha_1 x_2^2 + \alpha_2 \alpha_3 x_1^2 - \alpha_1 \alpha_2 x_3^2)}, \end{aligned}$$

are integrals of dET. We have: $K_\Psi(x) = K_{\Psi_0}(x) \oplus K_{\Psi_{12}}(x)$.

7. ZHUKOVSKI-VOLTERRA SYSTEM

The gyroscopic Zhukovski-Volterra (ZV) system is a generalization of the Euler top. It describes the free motion of a rigid body carrying an asymmetric rotor (gyrostat) [28]. Equations of motion of the ZV system read

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3 + \beta_3 x_2 - \beta_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1 + \beta_1 x_3 - \beta_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2 + \beta_2 x_1 - \beta_1 x_2, \end{cases} \quad (35)$$

with α_i, β_i being real parameters of the system. For $(\beta_1, \beta_2, \beta_3) = (0, 0, 0)$, the flow (35) reduces to the Euler top (21). The ZV system is (Liouville and algebraically) integrable under the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \quad (36)$$

It can be explicitly integrated in terms of elliptic functions, see [28] and also [3] for a more recent exposition. The following quantities are integrals of motion of the ZV system:

$$\begin{aligned} H_1 &= \alpha_2 x_3^2 - \alpha_3 x_2^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3), \\ H_2 &= \alpha_3 x_1^2 - \alpha_1 x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3), \\ H_3 &= \alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3). \end{aligned} \quad (37)$$

Clearly, only two of them are functionally independent because of $\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3 = 0$. Note that

$$H_2 - H_1 = \alpha_3 C, \quad H_3 - H_2 = \alpha_1 C, \quad H_1 - H_3 = \alpha_2 C,$$

with $C = x_1^2 + x_2^2 + x_3^2$.

As in the Euler case, the Wronskians of the coordinates x_j admit quadratic expressions with coefficients dependent on the integrals of motion:

$$\begin{cases} \dot{x}_2 x_3 - x_2 \dot{x}_3 = H_1 x_1 + x_1(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_1 C, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 = H_2 x_2 + x_2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_2 C, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3 + x_3(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_3 C. \end{cases} \quad (38)$$

The HK discretization of the ZV system is:

$$\begin{cases} \tilde{x}_1 - x_1 = \epsilon \alpha_1 (\tilde{x}_2 x_3 + x_2 \tilde{x}_3) + \epsilon \beta_3 (\tilde{x}_2 + x_2) - \epsilon \beta_2 (\tilde{x}_3 + x_3), \\ \tilde{x}_2 - x_2 = \epsilon \alpha_2 (\tilde{x}_3 x_1 + x_3 \tilde{x}_1) + \epsilon \beta_1 (\tilde{x}_3 + x_3) - \epsilon \beta_3 (\tilde{x}_1 + x_1), \\ \tilde{x}_3 - x_3 = \epsilon \alpha_3 (\tilde{x}_1 x_2 + x_1 \tilde{x}_2) + \epsilon \beta_2 (\tilde{x}_1 + x_1) - \epsilon \beta_1 (\tilde{x}_2 + x_2). \end{cases} \quad (39)$$

The map $f : x \mapsto \tilde{x}$ obtained by solving (39) for \tilde{x} is given by:

$$\tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(1 + \epsilon B)x,$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 & -\epsilon \alpha_1 x_3 & -\epsilon \alpha_1 x_2 \\ -\epsilon \alpha_2 x_3 & 1 & -\epsilon \alpha_2 x_1 \\ -\epsilon \alpha_3 x_2 & -\epsilon \alpha_3 x_1 & 1 \end{pmatrix} - \epsilon B, \quad B = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}.$$

We will call this map dZV. Formula (3) holds true for dZV, as for any HK discretization.

7.1. ZV system with two vanishing β_k 's. In the case when two out of three β_k 's vanish, say $\beta_2 = \beta_3 = 0$, the condition (36) is not necessary for integrability of the ZV system. The functions H_2 and H_3 as given in (37) (with $\beta_2 = \beta_3 = 0$) are in this case conserved quantities without any condition on α_k 's, while their linear combinations H_1 and C are given by

$$H_1 = -\frac{1}{\alpha_1}(\alpha_2 H_2 + \alpha_3 H_3) = \alpha_2 x_3^2 - \alpha_3 x_2^2 + 2\beta_1 \frac{\alpha_2 + \alpha_3}{\alpha_1} x_1,$$

$$C = \frac{1}{\alpha_1}(H_3 - H_2) = x_2^2 + x_3^2 - \frac{\alpha_2 + \alpha_3}{\alpha_1} x_1^2.$$

Wronskian relations (38) are replaced by

$$\begin{cases} \dot{x}_2 x_3 - x_2 \dot{x}_3 = H_1 x_1 - \beta_1 \frac{\alpha_2 + \alpha_3}{\alpha_1} x_1^2 + \beta_1 C, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 = H_2 x_2 + \beta_1 x_1 x_2, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3 + \beta_1 x_1 x_3. \end{cases} \quad (40)$$

The HK discretization of the ZV system with $\beta_2 = \beta_3 = 0$ turns out to possess two conserved quantities (without imposing condition (36)) and an invariant measure.

Proposition 18. *The functions*

$$H_2(\epsilon) = \frac{\alpha_3 x_1^2 - \alpha_1 x_3^2 - 2\beta_1 x_1 + \epsilon^2 \beta_1^2 \alpha_1 x_2^2}{1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2},$$

$$H_3(\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2\beta_1 x_1 - \epsilon^2 \beta_1^2 \alpha_1 x_3^2}{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2},$$

are conserved quantities of dZV with $\beta_2 = \beta_3 = 0$.

Proposition 19. *The map dZV with $\beta_2 = \beta_3 = 0$ possesses an invariant volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\phi(\tilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = (1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2)(1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)$.

The conserved quantities of Proposition 18 appear on the right-hand sides of the following relations which are the discrete versions of the Wronskian relations (40):

Proposition 20. *The following relations hold true for dZV with $\beta_2 = \beta_3 = 0$:*

$$\begin{cases} \tilde{x}_2 x_3 - x_2 \tilde{x}_3 = \epsilon c_1 (\tilde{x}_1 + x_1) + 2\epsilon c_2 \tilde{x}_1 x_1 + 2\epsilon c_3, \\ \tilde{x}_3 x_1 - x_3 \tilde{x}_1 = \epsilon H_2(\epsilon) (\tilde{x}_2 + x_2) + \epsilon \beta_1 (\tilde{x}_1 x_2 + x_1 \tilde{x}_2), \\ \tilde{x}_1 x_2 - x_1 \tilde{x}_2 = \epsilon H_3(\epsilon) (\tilde{x}_3 + x_3) + \epsilon \beta_1 (\tilde{x}_1 x_3 + x_1 \tilde{x}_3), \end{cases}$$

with

$$c_1 = -\frac{\alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon)}{\alpha_1 \Delta}, \quad c_2 = -\frac{\beta_1 (\alpha_2 + \alpha_3)}{\alpha_1 \Delta}, \quad c_3 = \frac{\beta_1 (H_3(\epsilon) - H_2(\epsilon))}{\alpha_1 \Delta},$$

$$\Delta = 1 + \epsilon^4 (\alpha_2 H_3(\epsilon) - \beta_1^2) (\alpha_3 H_2(\epsilon) + \beta_1^2).$$

Next, we describe the HK bases found in this case.

Proposition 21.

(a) *The set $\Phi = (x_1^2, x_2^2, x_3^2, x_1, 1)$ is a HK basis for dZV with $\beta_2 = \beta_3 = 0$, with $\dim K_\Phi(x) = 2$. Any orbit of dZV with $\beta_2 = \beta_3 = 0$ is thus confined to the intersection of two quadrics in \mathbb{R}^3 .*

(b) *The set $\Phi_0 = (x_1^2, x_2^2, x_3^2, 1)$ is a HK basis for dZV with $\beta_2 = \beta_3 = 0$, with $\dim K_{\Phi_0}(x) = 1$. At each point $x \in \mathbb{R}^3$ we have: $K_{\Phi_0}(x) = [-1 : d_2 : d_3 : d_4]$, where*

$$d_2 = \frac{\alpha_1}{\alpha_2 + \alpha_3} (1 - \epsilon^2 \beta_1^2 - \epsilon^2 \alpha_3 H_2(\epsilon)), \quad d_3 = \frac{\alpha_1}{\alpha_2 + \alpha_3} (1 - \epsilon^2 \beta_1^2 + \epsilon^2 \alpha_2 H_3(\epsilon)),$$

$$d_4 = \frac{1}{\alpha_2 + \alpha_3} (H_2(\epsilon) - H_3(\epsilon)).$$

(c) *The set $\Phi_{23} = (x_2^2, x_3^2, x_1, 1)$ is a HK basis for dZV with $\beta_2 = \beta_3 = 0$, with $\dim K_{\Phi_{23}}(x) = 1$. At each point $x \in \mathbb{R}^3$ we have: $K_{\Phi_{23}}(x) = [c_1 : c_2 : c_3 : c_4]$, where*

$$c_1 = \alpha_1 (\alpha_3 + \epsilon^2 \beta_1^2 \alpha_2 + \epsilon^2 \alpha_2 \alpha_3 H_2(\epsilon)), \quad c_2 = -\alpha_1 (\alpha_2 + \epsilon^2 \beta_1^2 \alpha_3 - \epsilon^2 \alpha_2 \alpha_3 H_3(\epsilon)),$$

$$c_3 = -2\beta_1 (\alpha_2 + \alpha_3), \quad c_4 = -(\alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon)).$$

Unlike the case of dET, we see that here a HK basis with a one dimensional null-space already provides more than one independent integral of motion.

“Bilinear” versions of the above HK bases also exist:

Proposition 22. *The set $\Psi = (x_1 \tilde{x}_1, x_2 \tilde{x}_2, x_3 \tilde{x}_3, x_1 + \tilde{x}_1, 1)$ is a HK basis for dZV with $\beta_2 = \beta_3 = 0$, with $\dim K_\Psi(x) = 2$. The sets*

$$\Psi_0 = (x_1 \tilde{x}_1, x_2 \tilde{x}_2, x_3 \tilde{x}_3, 1) \quad \text{and} \quad \Psi_{23} = (x_2 \tilde{x}_2, x_3 \tilde{x}_3, x_1 + \tilde{x}_1, 1)$$

are HK bases with one-dimensional null-spaces.

The following statement is a starting point towards an explicit integration of the map dZV with $\beta_2 = \beta_3 = 0$ in terms of elliptic functions.

Proposition 23. *The component x_1 of the solution of the difference equations (39) satisfies a relation of the type*

$$P(x_1, \tilde{x}_1) = p_0 x_1^2 \tilde{x}_1^2 + p_1 x_1 \tilde{x}_1 (x_1 + \tilde{x}_1) + p_2 (x_1^2 + \tilde{x}_1^2) + p_3 x_1 \tilde{x}_1 + p_4 (x_1 + \tilde{x}_1) + p_5 = 0,$$

coefficients of the biquadratic polynomial P being conserved quantities of dZV with $\beta_2 = \beta_3 = 0$.

Proof is parallel to that of Proposition 16.

7.2. ZV system with one vanishing β_k . In the case $\beta_3 = 0$ (say) and generic values of other parameters, the ZV system has only one integral H_3 and is therefore non-integrable. One of the Wronskian relations holds true in this general situation:

$$\dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3 + \beta_1 x_1 x_3 + \beta_2 x_2 x_3. \quad (41)$$

Under condition (36), the ZV system becomes integrable, with all the results formulated in the general case.

Similarly, the map dZV with $\beta_3 = 0$ and generic values of other parameters possesses one conserved quantity:

$$H_3(\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2) - \epsilon^2(\beta_1^2 \alpha_1 + \beta_2^2 \alpha_2) x_3^2}{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2}.$$

Clearly, this fact can be re-formulated as the existence of a HK basis $\Phi = (x_1^2, x_2^2, x_3^2, x_1, x_2, 1)$ with $\dim K_\Phi = 1$. The Wronskian relation (41) possesses a decent discretization:

$$\tilde{x}_1 x_2 - x_1 \tilde{x}_2 = \epsilon H_3(\epsilon)(x_3 + \tilde{x}_3) + \epsilon \beta_1(\tilde{x}_1 x_3 + x_1 \tilde{x}_3) + \epsilon \beta_2(\tilde{x}_2 x_3 + x_2 \tilde{x}_3).$$

However, it seems that the map dZV with $\beta_3 = 0$ does not acquire an additional integral of motion under condition (36). It might be conjectured that in order to assure the integrability of the dZV map with $\beta_3 = 0$, its other parameters have to satisfy some relation which is an $O(\epsilon)$ -deformation of (36).

7.3. ZV system with all β_k 's non-vanishing. Numerical experiments indicate non-integrability for the map (39) with non-vanishing β_k 's, even under condition (36). Nevertheless, some other relation between the parameters might yield integrability. In this connection we notice that the map dZV with $(\alpha_1, \alpha_2, \alpha_3) = (\alpha, -\alpha, 0)$ admits a polynomial conserved quantity

$$H = -\alpha x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \epsilon^2 \alpha (\beta_2 x_1 - \beta_1 x_2)^2.$$

8. VOLTERRA CHAIN

8.1. Periodic Volterra chain with $N = 3$ particles. Equations of motion of the periodic Volterra chain with three particles (VC₃, for short):

$$\begin{cases} \dot{x}_1 = x_1(x_2 - x_3), \\ \dot{x}_2 = x_2(x_3 - x_1), \\ \dot{x}_3 = x_3(x_1 - x_2). \end{cases} \quad (42)$$

This system is Liouville and algebraically integrable, with the following two independent integrals of motion:

$$H_1 = x_1 + x_2 + x_3, \quad H_2 = x_1 x_2 x_3.$$

There hold the following Wronskian relations:

$$\dot{x}_i x_j - x_i \dot{x}_j = H_1 x_i x_j - 3H_2. \quad (43)$$

Eliminating x_j, x_k from equation of motion for x_i with the help of integrals of motion, one arrives at

$$\dot{x}_i^2 = x_i^2(x_i - H_1)^2 - 4H_2 x_i,$$

which yields a solution in terms of elliptic functions.

The HK discretization of system (42) (with the time step 2ϵ) is:

$$\begin{cases} \tilde{x}_1 - x_1 = \epsilon x_1(\tilde{x}_2 - \tilde{x}_3) + \epsilon \tilde{x}_1(x_2 - x_3), \\ \tilde{x}_2 - x_2 = \epsilon x_2(\tilde{x}_3 - \tilde{x}_1) + \epsilon \tilde{x}_2(x_3 - x_1), \\ \tilde{x}_3 - x_3 = \epsilon x_3(\tilde{x}_1 - \tilde{x}_2) + \epsilon \tilde{x}_3(x_1 - x_2). \end{cases} \quad (44)$$

The map $f : x \mapsto \tilde{x}$ obtained by solving (44) for \tilde{x} is given by:

$$\tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x,$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 + \epsilon(x_3 - x_2) & -\epsilon x_1 & \epsilon x_1 \\ \epsilon x_2 & 1 + \epsilon(x_1 - x_3) & -\epsilon x_2 \\ -\epsilon x_3 & \epsilon x_3 & 1 + \epsilon(x_2 - x_1) \end{pmatrix}.$$

Explicitly:

$$\tilde{x}_i = x_i \frac{1 + 2\epsilon(x_j - x_k) + \epsilon^2((x_j + x_k)^2 - x_i^2)}{1 - \epsilon^2(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1)}. \quad (45)$$

This map will be called dVC_3 . From Proposition 2 there follows immediately:

Proposition 24. *The map dVC_3 possesses an invariant volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\phi(\tilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = x_1x_2x_3$.

Concerning integrability of dVC_3 , we note first of all that H_1 is an obvious conserved quantity. The second one is most easily obtained from the following discretization of Wronskian relations (43).

Proposition 25. *For the map dVC_3 , the following relations hold:*

$$\tilde{x}_ix_j - x_i\tilde{x}_j = \epsilon H_1(\tilde{x}_ix_j + x_i\tilde{x}_j) - 6\epsilon H_2(\epsilon) \left(1 - \frac{1}{3}\epsilon^2 H_1^2\right), \quad (46)$$

where $H_2(\epsilon)$ is a conserved quantity, given by

$$H_2(\epsilon) = \frac{x_1x_2x_3}{1 - \epsilon^2(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1)}. \quad (47)$$

Proof. Define $H_2(\epsilon)$ by eq. (46). It is easily computed with explicit formulas (45). The result given by (47) is a manifestly even function of ϵ and therefore an integral of motion. \square

Proposition 26.

(a) *The set $\Phi_{ij} = (x_ix_j(x_i + x_j), x_i^2 + x_j^2, x_ix_j, x_i + x_j, 1)$ is a HK basis for the map dVC_3 with $\dim K_{\Phi_{ij}}(x) = 1$. In other words, the pairs (x_i, x_j) lie on a cubic curve*

$$P(x_i, x_j) = p_0x_ix_j(x_i + x_j) + p_1(x_i^2 + x_j^2) + p_2x_ix_j + p_3(x_i + x_j) + p_4 = 0,$$

whose coefficients p_m are constant (expressed through integrals of motion).

(b) *The set $\Psi_i = (x_i^2\tilde{x}_i^2, x_i\tilde{x}_i(x_i + \tilde{x}_i), x_i^2 + \tilde{x}_i^2, x_i\tilde{x}_i, x_i + \tilde{x}_i, 1)$ is a HK basis for the map dVC_3 with $\dim K_{\Psi_i}(x) = 1$. In other words, the pairs (x_i, \tilde{x}_i) lie on a symmetric biquadratic curve with constant coefficients (which can be expressed through integrals of motion).*

Proof. Statement (a) follows by eliminating x_k from (47) via $x_k = H_1 - x_i - x_j$. Statement (b) is obtained with the help of MAPLE; it implies that x_i as functions of t are elliptic functions of degree 2 (i.e., with two poles within one parallelogram of periods). \square

8.2. Periodic Volterra chain with $N = 4$ particles. Equations of motion of VC_4 are:

$$\begin{cases} \dot{x}_1 = x_1(x_2 - x_4), \\ \dot{x}_2 = x_2(x_3 - x_1), \\ \dot{x}_3 = x_3(x_4 - x_2), \\ \dot{x}_4 = x_4(x_1 - x_3). \end{cases}$$

This system possesses three obvious integrals of motion: $H_1 = x_1 + x_2 + x_3 + x_4$, $H_2 = x_1x_3$, and $H_3 = x_2x_4$. One easily finds that x_1, x_3 satisfy the differential equation

$$\dot{x}_1^2 = (x_1^2 - H_1x_1 + H_2)^2 - 4H_3x_1^2,$$

while x_2, x_4 satisfy a similar equation with $H_2 \leftrightarrow H_3$. This immediately leads to solution in terms of elliptic functions. There are two types of Wronskian relations:

$$\dot{x}_1 x_3 - x_1 \dot{x}_3 = 2H_2(x_2 - x_4), \quad \dot{x}_2 x_4 - x_2 \dot{x}_4 = 2H_3(x_3 - x_1), \quad (48)$$

and

$$\dot{x}_1 x_2 - x_1 \dot{x}_2 = H_1 x_1 x_2 - 2H_2 x_2 - 2H_3 x_1.$$

HK discretization (denoted by dVC_4):

$$\begin{cases} \tilde{x}_1 - x_1 = \epsilon x_1(\tilde{x}_2 - \tilde{x}_4) + \epsilon \tilde{x}_1(x_2 - x_4), \\ \tilde{x}_2 - x_2 = \epsilon x_2(\tilde{x}_3 - \tilde{x}_1) + \epsilon \tilde{x}_2(x_3 - x_1), \\ \tilde{x}_3 - x_3 = \epsilon x_3(\tilde{x}_4 - \tilde{x}_2) + \epsilon \tilde{x}_3(x_4 - x_2), \\ \tilde{x}_4 - x_4 = \epsilon x_4(\tilde{x}_1 - \tilde{x}_3) + \epsilon \tilde{x}_4(x_1 - x_4). \end{cases}$$

It possesses an obvious integral $H_1 = x_1 + x_2 + x_3 + x_4$.

Proposition 27. *For the map dVC_4 , the following natural discretization of eqs. (48) holds:*

$$\begin{aligned} \tilde{x}_1 x_3 - x_1 \tilde{x}_3 &= 2\epsilon H_2(\epsilon)(x_2 + \tilde{x}_2 - x_4 - \tilde{x}_4), \\ \tilde{x}_2 x_4 - x_2 \tilde{x}_4 &= 2\epsilon H_3(\epsilon)(x_1 + \tilde{x}_1 - x_3 - \tilde{x}_3), \end{aligned}$$

with the conserved quantities

$$H_2(\epsilon) = \frac{x_1 x_3}{1 - \epsilon^2(x_2 - x_4)^2}, \quad H_3(\epsilon) = \frac{x_2 x_4}{1 - \epsilon^2(x_1 - x_3)^2}.$$

Proof. This can be shown directly; the fact that $H_2(\epsilon), H_3(\epsilon)$ are even functions of ϵ , assures that they are conserved quantities. One can also show this immediately from equations of motion: for instance, multiplying the equations

$$\frac{\tilde{x}_1}{1 + \epsilon(\tilde{x}_2 - \tilde{x}_4)} = \frac{x_1}{1 - \epsilon(x_2 - x_4)}, \quad \frac{\tilde{x}_3}{1 - \epsilon(\tilde{x}_2 - \tilde{x}_4)} = \frac{x_3}{1 + \epsilon(x_2 - x_4)},$$

shows that $H_2(\epsilon)$ is a conserved quantity. \square

Proposition 28.

(a) *For the iterates of map dVC_4 , the pairs (x_1, x_2) lie on a quartic curve whose coefficients are constant (expressed through integrals of motion).*

(b) *The pairs (x_i, \tilde{x}_i) lie on a biquartic curve of genus 1 with constant coefficients (which can be expressed through integrals of motion).*

Proof. Statement (a) follows by eliminating x_3, x_4 from integrals $H_1, H_2(\epsilon), H_3(\epsilon)$. Statement (b) is obtained with the help of MAPLE; it implies that x_i as functions of t are elliptic functions of degree 4 (i.e., with four poles within one parallelogram of periods). \square

Note that the reduction $x_4 = 0$ of the periodic Volterra chain with $N = 4$ leads to the open-end Volterra chain with $N = 3$ particles.

9. DRESSING CHAIN ($N = 3$)

The three-dimensional dressing chain (DC₃, for short) is described by the following system of quadratic ordinary equations [27]:

$$\begin{cases} \dot{x}_1 = x_3^2 - x_2^2 + \alpha_3 - \alpha_2, \\ \dot{x}_2 = x_1^2 - x_3^2 + \alpha_1 - \alpha_3, \\ \dot{x}_3 = x_2^2 - x_1^2 + \alpha_2 - \alpha_1, \end{cases} \quad (49)$$

with real parameters α_i . The system (49) is (Liouville and algebraically) integrable. The following quantities are integrals of motion:

$$\begin{aligned} I_1 &= x_1 + x_2 + x_3, \\ I_2 &= (x_1 + x_2)(x_2 + x_3)(x_3 + x_1) - \alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_3. \end{aligned}$$

Sometimes it is more convenient to use the following integral instead of I_2 :

$$H_2 = x_1^3 + x_2^3 + x_3^3 + 3\alpha_1 x_1 + 3\alpha_2 x_2 + 3\alpha_3 x_3 = I_1^3 - 3I_2.$$

There hold the following Wronskian relations:

$$\dot{x}_i x_j - x_i \dot{x}_j = I_1 x_k^2 + 2(\alpha_i - \alpha_k)x_i + 2(\alpha_j - \alpha_k)x_j + 3\alpha_k I_1 - H_2. \quad (50)$$

Excluding x_j, x_k from equations of motion for x_i with the help of integrals of motion, one arrives at

$$\dot{x}_i^2 = x_i^4 + 6a_2 x_i^2 + 4a_3 x_i + a_4, \quad (51)$$

with

$$\begin{aligned} a_2 &= -\frac{1}{3}(I_1^2 + \alpha_j + \alpha_k - 2\alpha_i), & a_3 &= (\alpha_j + \alpha_k - \alpha_i)I_1 + I_2, \\ a_4 &= I_1^4 - 2(\alpha_j + \alpha_k)I_1^2 + (\alpha_j - \alpha_k)^2 - 4I_1 I_2. \end{aligned}$$

All three elliptic curves corresponding to (51) with $i = 1, 2, 3$, have equal Weierstrass invariants (expressed through the parameters α_i and the integrals of motion):

$$g_2 = a_4 + 3a_2^2, \quad g_3 = a_2 a_4 - a_2^3 - a_3^2, \quad \text{so that} \quad a_3^2 = -4a_2^3 + g_2 a_2 - g_3.$$

The coefficients in (51) can be thus parametrized in terms of the Weierstrass elliptic function with the invariants g_2, g_3 as follows: $a_2 = -\wp(A_i)$, $a_3 = \wp'(A_i)$, so that $a_4 = g_2 - 3\wp^2(A_i)$. One can show that $A_1 + A_2 + A_3 = 0$ (modulo the period lattice), so that one can introduce B_i , defined up to a common additive shift, through $A_i = B_i - B_{i+1}$. The solution of the dressing chain DC₃ is then given as

$$x_i(t) = \zeta(t - B_{i+1}) - \zeta(t - B_i) - \zeta(B_i - B_{i+1}).$$

The HK discretization of system (49) is:

$$\begin{cases} \tilde{x}_1 - x_1 = \epsilon(\tilde{x}_3 x_3 - \tilde{x}_2 x_2 + \alpha_3 - \alpha_2), \\ \tilde{x}_2 - x_2 = \epsilon(\tilde{x}_1 x_1 - \tilde{x}_3 x_3 + \alpha_1 - \alpha_3), \\ \tilde{x}_3 - x_3 = \epsilon(\tilde{x}_2 x_2 - \tilde{x}_1 x_1 + \alpha_2 - \alpha_1). \end{cases} \quad (52)$$

The map $f : x \mapsto \tilde{x}$ obtained by solving (52) for \tilde{x} is given by:

$$\tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(x + \epsilon c),$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 & \epsilon x_2 & -\epsilon x_3 \\ -\epsilon x_1 & 1 & \epsilon x_3 \\ \epsilon x_1 & -\epsilon x_2 & 1 \end{pmatrix}, \quad c = (\alpha_3 - \alpha_2, \alpha_1 - \alpha_3, \alpha_2 - \alpha_1)^T.$$

Explicitly:

$$\tilde{x}_i = \frac{x_i + \epsilon(x_k^2 - x_j^2 + \alpha_k - \alpha_j) + \epsilon^2(I_1 x_j x_k + (\alpha_k - \alpha_i)x_j + (\alpha_j - \alpha_i)x_k)}{1 + \epsilon^2(x_1 x_2 + x_2 x_3 + x_3 x_1)}. \quad (53)$$

This map will be called dDC₃. From Proposition 1 there follows immediately:

Proposition 29. *The map dDC₃ possesses an invariant volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\phi(\tilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = 1 + \epsilon^2(x_1 x_2 + x_2 x_3 + x_3 x_1)$.

Concerning integrability of dDC₃, we note first of all that I_1 is an obvious conserved quantity. The second one is most easily obtained from the following discretization of Wronskian relations (50).

Proposition 30. *For the map dDC₃, the following relations hold:*

$$\frac{\tilde{x}_i x_j - x_i \tilde{x}_j}{\epsilon} = I_1 x_k \tilde{x}_k + (\alpha_i - \alpha_k)(x_i + \tilde{x}_i) + (\alpha_j - \alpha_k)(x_j + \tilde{x}_j) + 3\alpha_k I_1 - H_2(\epsilon), \quad (54)$$

where $H_2(\epsilon)$ is a conserved quantity, given by

$$H_2(\epsilon) = \frac{H_2 + \epsilon^2 G_2}{1 + \epsilon^2(x_1 x_2 + x_2 x_3 + x_3 x_1)}, \quad (55)$$

where

$$\begin{aligned} G_2 = & I_1^2 x_1 x_2 x_3 + (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)(x_1 x_2 + x_2 x_3 + x_3 x_1) \\ & + 2I_1(\alpha_1 x_2 x_3 + \alpha_2 x_3 x_1 + \alpha_3 x_1 x_2) \\ & - (\alpha_2 - \alpha_3)^2 x_1 - (\alpha_3 - \alpha_1)^2 x_2 - (\alpha_1 - \alpha_2)^2 x_3. \end{aligned}$$

Proof. Define $H_2(\epsilon)$ by eq. (54). It is easily computed with explicit formulas (53). The result given by (55) is a manifestly even function of ϵ and therefore an integral of motion. \square

Proposition 31.

(a) *The set $\Phi_{ij} = (x_i^3, x_j^3, x_i x_j(x_i + x_j), x_i^2, x_i x_j, x_j^2, x_i, x_j, 1)$ is a HK basis for the map dDC₃ with $\dim K_{\Phi_{ij}}(x) = 1$. In other words, the pairs x_i, x_j satisfy equations of degree 3,*

$$P_{ij}(x_i, x_j) = p_0 x_i^3 + p_1 x_j^3 + p_2 x_i x_j(x_i + x_j) + p_3 x_i^2 + p_4 x_i x_j + p_5 x_j^2 + p_6 x_i + p_7 x_j + p_8 = 0,$$

whose coefficients $p_m = p_m^{(ij)}$ are constant (expressed through parameters α_k and integrals of motion).

(b) *The set $\Psi_i = (x_i^m \tilde{x}_i^n)_{m,n=0}^3$ is a HK basis for the map dDC₃ with $\dim K_{\Psi_i}(x) = 1$. In other words, the pairs x_i, \tilde{x}_i lie on bicubic curves of genus 1:*

$$Q_i(x_i, \tilde{x}_i) = \sum_{m,n=0}^3 q_{mn} x_i^m \tilde{x}_i^n = 0,$$

whose coefficients $q_{mn} = q_{mn}^{(i)}$ are constant (expressed through parameters α_k and integrals of motion). Moreover, $q_{13} = q_{31}$.

Proof. Statement (a) follows by eliminating x_k from (55) via $x_k = I_1 - x_i - x_j$. Statement (b) is obtained with the help of MAPLE. One can also show that these bicubic curves are of genus 1, so that x_i as functions of t are elliptic functions of degree 3 (i.e., with three poles within one parallelogram of periods). \square

10. COUPLED EULER TOPS

In [6] a remarkable mechanical system was introduced, which can be interpreted as a chain of coupled three-dimensional Euler tops. The differential equations governing the system are given by:

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_{2j} = \alpha_{3j-1} x_{2j-1} x_{2j+1}, \\ \dot{x}_{2j+1} = \alpha_{3j} x_{2j} x_{2j-1} + \alpha_{3j+1} x_{2j+2} x_{2j+3}, \\ \dot{x}_{2j+2} = \alpha_{3j+2} x_{2j+1} x_{2j+3}, \\ \dot{x}_{2N+1} = \alpha_{3N} x_{2N} x_{2N-1}, \end{cases} \quad (56)$$

with real parameters α_i . Each triple of variables $(x_{2j-1}, x_{2j}, x_{2j+1})$ can be considered as a 3D Euler top, coupled with the neighboring triple $(x_{2j+1}, x_{2j+2}, x_{2j+3})$ via the variable x_{2j+1} . We will denote system (56) by CET_N . It has $N + 1$ independent conserved quantities:

$$\begin{aligned} H_1 &= \alpha_2 x_1^2 - \alpha_1 x_2^2, \\ H_j &= \alpha_{3j-3} \alpha_{3j-1} x_{2j-2}^2 - \alpha_{3j-4} \alpha_{3j-1} x_{2j-1}^2 + \alpha_{3j-4} \alpha_{3j-2} x_{2j}^2, \quad 2 \leq j \leq N, \\ H_{N+1} &= \alpha_{3N} x_{2N}^2 - \alpha_{3N-1} x_{2N+1}^2. \end{aligned}$$

Nothing is known about the possible Hamiltonian formulation of this system, and therefore about its integrability in the Liouville-Arnold sense.

For $N = 1$ system (56) reduces to the usual Euler top (21). We will consider in detail the HK discretization of the system CET_2 given by

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2 + \alpha_4 x_4 x_5, \\ \dot{x}_4 = \alpha_5 x_5 x_3, \\ \dot{x}_5 = \alpha_6 x_3 x_4. \end{cases} \quad (57)$$

It can be interpreted as two Euler tops, described by the two sets of variables (x_1, x_2, x_3) and (x_3, x_4, x_5) , respectively, coupled via the variable x_3 . It has three independent integrals of motion:

$$\begin{aligned} H_1 &= \alpha_2 x_1^2 - \alpha_1 x_2^2, & H_3 &= \alpha_6 x_4^2 - \alpha_5 x_5^2, \\ H_2 &= \alpha_3 \alpha_5 x_2^2 - \alpha_2 \alpha_5 x_3^2 + \alpha_2 \alpha_4 x_4^2, \end{aligned}$$

and it can be solved in terms of elliptic functions. We will be mainly interested in its particular case which is superintegrable.

Proposition 32. *If the coefficients α_i satisfy the following condition,*

$$\alpha_1\alpha_2 = \alpha_5\alpha_6, \quad (58)$$

then the system CET_2 is superintegrable: it has two additional integrals,

$$H_4 = \alpha_5x_2x_5 - \alpha_2x_1x_4, \quad H_5 = \alpha_5x_1x_5 - \alpha_1x_2x_4,$$

and among the functions H_1, \dots, H_5 there are four independent ones.

In this case, the variable x_3 satisfies the following differential equation:

$$\dot{x}_3^2 = \left(x_3^2 + \frac{H_2}{\alpha_2\alpha_5} \right) \left(\alpha_1\alpha_2x_3^2 + \frac{\alpha_1}{\alpha_5}H_2 + \alpha_3H_1 - \alpha_4H_3 \right) - \frac{\alpha_3\alpha_4}{\alpha_2\alpha_5}H_4^2, \quad (59)$$

so that its time evolution is described by an elliptic function of degree 2.

The HK discretization of the system CET_2 reads:

$$\begin{cases} \tilde{x}_1 - x_1 = \epsilon\alpha_1(\tilde{x}_2x_3 + x_2\tilde{x}_3), \\ \tilde{x}_2 - x_2 = \epsilon\alpha_2(\tilde{x}_3x_1 + x_3\tilde{x}_1), \\ \tilde{x}_3 - x_3 = \epsilon\alpha_3(\tilde{x}_1x_2 + x_1\tilde{x}_2) + \epsilon\alpha_4(\tilde{x}_4x_5 + x_4\tilde{x}_5), \\ \tilde{x}_4 - x_4 = \epsilon\alpha_5(\tilde{x}_5x_3 + x_5\tilde{x}_3), \\ \tilde{x}_5 - x_5 = \epsilon\alpha_6(\tilde{x}_3x_4 + x_3\tilde{x}_4). \end{cases} \quad (60)$$

The map $f : x \mapsto \tilde{x}$ obtained by solving (60) for \tilde{x} is given by:

$$\tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x,$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 & \epsilon\alpha_1x_3 & \epsilon\alpha_1x_2 & 0 & 0 \\ \epsilon\alpha_2x_3 & 1 & \epsilon\alpha_2x_1 & 0 & 0 \\ \epsilon\alpha_3x_2 & \epsilon\alpha_3x_1 & 1 & \epsilon\alpha_4x_5 & \epsilon\alpha_4x_4 \\ 0 & 0 & \epsilon\alpha_5x_5 & 1 & \epsilon\alpha_5x_3 \\ 0 & 0 & \epsilon\alpha_6x_4 & \epsilon\alpha_6x_3 & 1 \end{pmatrix}.$$

This map will be called $dCET_2$ in the sequel.

Proposition 33. *The functions*

$$H_1(\epsilon) = \frac{\alpha_2x_1^2 - \alpha_1x_2^2}{1 - \epsilon^2\alpha_1\alpha_2x_3^2}, \quad H_3(\epsilon) = \frac{\alpha_6x_4^2 - \alpha_5x_5^2}{1 - \epsilon^2\alpha_5\alpha_6x_3^2},$$

are conserved quantities of the map $dCET_2$.

Proposition 34 gives only two independent integrals of motion for the map $dCET_2$. Numerical experiments indicate that the third integral does not exist in general. The situation is different under condition (58). Note that in this case the denominators of the integrals $H_1(\epsilon)$ and $H_3(\epsilon)$ coincide.

Proposition 34. *If condition (58) holds, then the map $dCET_2$ has in addition to $H_1(\epsilon)$ and $H_3(\epsilon)$ also the following conserved quantities*

$$H_2(\epsilon) = \frac{\alpha_3\alpha_5x_2^2 - \alpha_2\alpha_5x_3^2 + \alpha_2\alpha_4x_4^2}{1 - \epsilon^2\alpha_1\alpha_2x_3^2},$$

$$H_4(\epsilon) = \frac{\alpha_5x_2x_5 - \alpha_2x_1x_4}{1 - \epsilon^2\alpha_1\alpha_2x_3^2}, \quad H_5(\epsilon) = \frac{\alpha_5x_1x_5 - \alpha_1x_2x_4}{1 - \epsilon^2\alpha_1\alpha_2x_3^2}.$$

There are four independent functions among $H_1(\epsilon), \dots, H_5(\epsilon)$.

We now present HK bases for the map dCET_2 .

Proposition 35. *Under condition (58), the map dCET_2 has the following HK bases.*

(a) *The set $\Phi = (x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, 1)$ is a HK basis with $\dim K_\Phi(x) = 3$.*

(b) *The sets $\Phi_1 = (x_1^2, x_2^2, x_3^2, 1)$ and $\Phi_2 = (x_3^2, x_4^2, x_5^2, 1)$ are HK bases with one-dimensional null-spaces. At each point $x \in \mathbb{R}^5$ we have: $K_{\Phi_1}(x) = [e_1 : e_2 : \epsilon^2 \alpha_1 \alpha_2 : -1]$ and $K_{\Phi_2}(x) = [\epsilon^2 \alpha_1 \alpha_2 : f_4 : f_5 : -1]$. The functions e_i and f_i are conserved quantities given by*

$$e_1 = \frac{\alpha_2(1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)}{\alpha_2 x_1^2 - \alpha_1 x_2^2}, \quad e_2 = -\frac{\alpha_1(1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)}{\alpha_2 x_1^2 - \alpha_1 x_2^2},$$

and

$$f_4 = \frac{\alpha_5(1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)}{\alpha_5 x_4^2 - \alpha_4 x_5^2}, \quad f_5 = -\frac{\alpha_4(1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)}{\alpha_5 x_4^2 - \alpha_4 x_5^2}.$$

The set $\Phi_3 = (x_1^2, x_2^2, x_3^2, x_4^2)$ is a HK basis with a one-dimensional null-space. At each point $x \in \mathbb{R}^5$ we have: $K_{\Phi_3}(x) = [g_1 : g_2 : \alpha_5 : -\alpha_4]$. The functions g_i are conserved quantities given by

$$g_1 = \frac{\alpha_3 \alpha_5 x_1^2 - \alpha_2 \alpha_5 x_3^2 + \alpha_2 \alpha_4 x_4^2}{\alpha_2 x_1^2 - \alpha_1 x_2^2}, \quad g_2 = -\frac{\alpha_3 \alpha_5 x_1^2 - \alpha_1 \alpha_5 x_3^2 + \alpha_1 \alpha_4 x_4^2}{\alpha_2 x_1^2 - \alpha_1 x_2^2}.$$

Similar claim hold for the sets $(x_1^2, x_2^2, x_3^2, x_5^2)$, $(x_1^2, x_3^2, x_4^2, x_5^2)$, and $(x_2^2, x_3^2, x_4^2, x_5^2)$.

(c) *The set $\Psi = (x_1, x_2, x_3, x_4, x_5)$ is a HK basis with $\dim K_\Psi(x) = 2$.*

(d) *The sets $\Psi_1 = (x_1, x_2, x_4)$, $\Psi_2 = (x_1, x_2, x_5)$ are HK bases with one-dimensional null-spaces. At each point $x \in \mathbb{R}^5$ we have: $K_{\Psi_1}(x) = [c_1 : c_2 : -1]$, $K_{\Psi_2}(x) = [d_1 : d_2 : -1]$. The functions c_1, c_2 and d_1, d_2 are conserved quantities given by*

$$c_1 = \frac{\alpha_2 x_1 x_4 - \alpha_5 x_2 x_5}{\alpha_2 x_1^2 - \alpha_1 x_2^2}, \quad c_2 = \frac{\alpha_5 x_1 x_5 - \alpha_1 x_2 x_4}{\alpha_2 x_1^2 - \alpha_1 x_2^2},$$

while

$$d_1 = \frac{\alpha_2}{\alpha_5} c_2, \quad d_2 = \frac{\alpha_1}{\alpha_5} c_1.$$

A similar claim holds for the sets (x_2, x_4, x_5) and (x_1, x_4, x_5) .

We see that the map dCET_2 under condition (58) possesses four functionally independent conserved quantities. It might look paradoxical that $\Phi \cup \Psi$ is a HK basis with a 5-dimensional null-space, thus imposing seemingly 5 restrictions on any orbit of the map, which would yield 0-dimensional invariant sets (instead of invariant curves in the continuous time case). The resolution of this paradox is that the five restrictions are functionally dependent on their common set (i.e., along any orbit). In other words, the HK basis $\Phi \cup \Psi$ is not regular. This is the first and the only instance of a non-regular HK basis in this paper.

The map dCET_2 possesses, in its (super)-integrable regime, an invariant volume form:

Proposition 36. *Under condition (58), the map dCET_2 preserves the following volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\phi(\tilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5}{\phi(x)},$$

with $\phi(x) = (1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)^3$.

In this regime, the solutions can be found in terms of elliptic functions, as the following statement shows.

Proposition 37. *Under condition (58), the component x_3 of any orbit of the map $dCET_2$ satisfies a relation of the type*

$$Q(x_3, \tilde{x}_3) = q_0 x_3^2 \tilde{x}_3^2 + q_1 x_3 \tilde{x}_3 (x_3 + \tilde{x}_3) + q_2 (x_3^2 + \tilde{x}_3^2) + q_3 x_3 \tilde{x}_3 + q_4 (x_3 + \tilde{x}_3) + q_5 = 0,$$

coefficients of the biquadratic polynomial Q being conserved quantities of $dCET_2$.

This statement is a proper discretization of eq. (59).

11. THREE WAVE SYSTEM

The three wave interaction system of ordinary differential equations is [2]:

$$\begin{cases} \dot{z}_1 = i\alpha_1 \bar{z}_2 \bar{z}_3, \\ \dot{z}_2 = i\alpha_2 \bar{z}_3 \bar{z}_1, \\ \dot{z}_3 = i\alpha_3 \bar{z}_1 \bar{z}_2. \end{cases} \quad (61)$$

Here $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, while the parameters α_i of the system are supposed to be real numbers. If (i, j, k) stands for any cyclic permutation of (123) , then we can write system (61) in the abbreviated form

$$\dot{z}_i = i\alpha_i \bar{z}_j \bar{z}_k, \quad (62)$$

Writing $z_i = x_i + iy_i$, $i = 1, 2, 3$, we put system (62) into the form

$$\begin{cases} \dot{x}_i = \alpha_i (x_j y_k + y_j x_k), \\ \dot{y}_i = \alpha_i (x_j x_k - y_j y_k). \end{cases} \quad (63)$$

System (63) is completely integrable and can be integrated in terms of elliptic functions. It has three independent integrals of motion: quadratic ones,

$$H_i = \alpha_j |z_k|^2 - \alpha_k |z_j|^2,$$

among which there are only two independent ones because of $\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3 = 0$, and a cubic one,

$$K = \frac{1}{2} (z_1 z_2 z_3 + \bar{z}_1 \bar{z}_2 \bar{z}_3) = \Re(z_1 z_2 z_3).$$

The HK discretization of system (62) reads

$$\tilde{z}_i - z_i = i\epsilon \alpha_i (\bar{z}_j \tilde{z}_k + \tilde{\bar{z}}_j \bar{z}_k),$$

or, in the real variables (x_i, y_i) ,

$$\begin{cases} \tilde{x}_i - x_i = \epsilon \alpha_i (x_j \tilde{y}_k + \tilde{x}_j y_k + y_j \tilde{x}_k + \tilde{y}_j x_k), \\ \tilde{y}_i - y_i = \epsilon \alpha_i (x_j \tilde{x}_k + \tilde{x}_j x_k - y_j \tilde{y}_k - \tilde{y}_j y_k). \end{cases}$$

In the matrix form, this can be put as

$$A(x, y, \epsilon) \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = f(x, y, \epsilon) = A^{-1}(x, y, \epsilon) \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$A(x, y, \epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 y_3 & -\epsilon\alpha_1 y_2 & 0 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 y_3 & 1 & -\epsilon\alpha_2 y_1 & -\epsilon\alpha_2 x_3 & 0 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 y_2 & -\epsilon\alpha_3 y_1 & 1 & -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 0 \\ 0 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 & 1 & \epsilon\alpha_1 y_3 & \epsilon\alpha_1 y_2 \\ -\epsilon\alpha_2 x_3 & 0 & -\epsilon\alpha_2 x_1 & \epsilon\alpha_2 y_3 & 1 & \epsilon\alpha_2 y_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 0 & \epsilon\alpha_3 y_2 & \epsilon\alpha_3 y_1 & 1 \end{pmatrix}.$$

The birational map $f : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ will be called d3W hereafter.

Proposition 38. *The map d3W has three independent conserved quantities, namely, any two of*

$$H_i(\epsilon) = \frac{\alpha_j |z_k|^2 - \alpha_k |z_j|^2}{1 - \epsilon^2 \alpha_j \alpha_k |z_i|^2},$$

supplied with any one of

$$K_i(\epsilon) = \frac{\Re(z_1 z_2 z_3)(1 - \epsilon^2 \alpha_k \alpha_i |z_j|^2)(1 - \epsilon^2 \alpha_i \alpha_j |z_k|^2)}{\Delta(z, \bar{z}, \epsilon)},$$

where

$$\begin{aligned} \Delta(z, \bar{z}, \epsilon) &= \det A(x, y, \epsilon) = 1 - 2\epsilon^2(\alpha_2 \alpha_3 |z_1|^2 + \alpha_3 \alpha_1 |z_2|^2 + \alpha_1 \alpha_2 |z_3|^2) \\ &\quad + \epsilon^4(\alpha_2 \alpha_3 |z_1|^2 + \alpha_3 \alpha_1 |z_2|^2 + \alpha_1 \alpha_2 |z_3|^2)^2 - 4\epsilon^6 \alpha_1^2 \alpha_2^2 \alpha_3^2 |z_1|^2 |z_2|^2 |z_3|^2. \end{aligned}$$

Proposition 39. *The map d3W possesses an invariant volume form:*

$$\det \frac{\partial \tilde{z}}{\partial z} = \frac{\phi(\tilde{z})}{\phi(z)} \Leftrightarrow f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3}{\phi(z)},$$

where $\phi(z) = \Delta(z, \bar{z}, \epsilon)$.

Next, we give the results on the HK bases for the map d3W, which yield a complete set of integrals of motion.

Proposition 40.

(a) *The sets $\Phi_i = (|z_j|^2, |z_k|^2, 1)$, $i = 1, 2, 3$, are HK bases for the map d3W with $\dim K_{\Phi_i}(z) = 1$. At each point $z \in \mathbb{C}^3$ there holds: $K_{\Phi_i}(z) = [d_1 : d_2 : -1]$, where the coefficients*

$$d_1(z) = \frac{\alpha_k(1 - \epsilon^2 \alpha_i \alpha_j |z_k|^2)}{\alpha_k |z_j|^2 - \alpha_j |z_k|^2}, \quad d_2(z) = -\frac{\alpha_j(1 - \epsilon^2 \alpha_k \alpha_i |z_j|^2)}{\alpha_k |z_j|^2 - \alpha_j |z_k|^2},$$

are integrals of motion of the map d3W. They are functionally dependent because of $\alpha_j d_1(z) + \alpha_k d_2(z) = \epsilon^2 \alpha_1 \alpha_2 \alpha_3$.

(b) *The sets $\Psi_i = (\Re(z_1 z_2 z_3), |z_i|^2, 1)$, $i = 1, 2, 3$, are HK bases for the map d3W with $\dim K_{\Psi_i}(z) = 1$. At each point $z \in \mathbb{C}^3$ there holds: $K_{\Psi_i}(z) = [e_1 : e_2 : -1]$, where the coefficients*

$$\begin{aligned} e_1(z) &= -\frac{\Delta(z, \bar{z}, \epsilon)}{\Re(z_1 z_2 z_3)(1 - \epsilon^2(-\alpha_j \alpha_k |z_i|^2 + \alpha_k \alpha_i |z_j|^2 + \alpha_i \alpha_j |z_k|^2))^2}, \\ e_2(z) &= \frac{4\alpha_j \alpha_k \epsilon^2(1 - \epsilon^2 \alpha_k \alpha_i |z_j|^2)(1 - \epsilon^2 \alpha_i \alpha_j |z_k|^2)}{(1 - \epsilon^2(-\alpha_j \alpha_k |z_i|^2 + \alpha_k \alpha_i |z_j|^2 + \alpha_i \alpha_j |z_k|^2))^2}, \end{aligned}$$

are independent integrals of motion of the map d3W.

12. LAGRANGE TOP

Lagrange top was the second integrable system, after Euler top, to which the HK discretization was successfully applied [12]. We reproduce and re-derive here the results of that paper, and add some new results.

Equations of motion of the Lagrange top are of the general Kirchhoff type:

$$\begin{cases} \dot{m} = m \times \nabla_m H + p \times \nabla_p H, \\ \dot{p} = p \times \nabla_m H, \end{cases} \quad (64)$$

where $m = (m_1, m_2, m_3)^T$ and $p = (p_1, p_2, p_3)^T$. Any Kirchhoff type system is Hamiltonian with the Hamilton function $H = H(m, p)$ with respect to the Lie-Poisson bracket on $\mathfrak{e}(3)^*$,

$$\{m_i, m_j\} = \epsilon_{ijk} m_k, \quad \{m_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0,$$

and admits the Hamilton function H and the Casimir functions

$$C_1 = p_1^2 + p_2^2 + p_3^2, \quad C_2 = m_1 p_1 + m_2 p_2 + m_3 p_3, \quad (65)$$

as integrals of motion. For the complete integrability of a Kirchhoff type system, it should admit a fourth independent integral of motion.

The Hamilton function of the Lagrange top (LT) is $H = H_1/2$, where

$$H_1 = m_1^2 + m_2^2 + \alpha m_3^2 + 2\gamma p_3.$$

Thus, equations of motion of LT read

$$\begin{cases} \dot{m}_1 = (\alpha - 1)m_2 m_3 + \gamma p_2, \\ \dot{m}_2 = (1 - \alpha)m_1 m_3 - \gamma p_1, \\ \dot{m}_3 = 0, \\ \dot{p}_1 = \alpha p_2 m_3 - p_3 m_2, \\ \dot{p}_2 = p_3 m_1 - \alpha p_1 m_3, \\ \dot{p}_3 = p_1 m_2 - p_2 m_1. \end{cases} \quad (66)$$

It follows immediately that the fourth integral of motion is simply $H_2 = m_3$. Traditionally, the explicit integration of the LT in terms of elliptic functions starts with the following observation: the component p_3 of the solution satisfies the differential equation

$$\dot{p}_3^2 = P_3(p_3), \quad (67)$$

with a cubic polynomial P_3 whose coefficients are expressed through integrals of motion:

$$P_3(p_3) = (H_1 - \alpha m_3^2 - 2\gamma p_3)(C_1 - p_3^2) - (C_2 - m_3 p_3)^2.$$

We mention also the following Wronskian relation which follows easily from equations of motion:

$$(\dot{m}_1 p_1 - m_1 \dot{p}_1) + (\dot{m}_2 p_2 - m_2 \dot{p}_2) + (2\alpha - 1)(\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0. \quad (68)$$

Applying the HK discretization scheme to eqs. (66), we obtain the following discrete system:

$$\begin{cases} \tilde{m}_1 - m_1 = \epsilon(\alpha - 1)(\tilde{m}_2 m_3 + m_2 \tilde{m}_3) + \epsilon\gamma(p_2 + \tilde{p}_2), \\ \tilde{m}_2 - m_2 = \epsilon(1 - \alpha)(\tilde{m}_1 m_3 + m_1 \tilde{m}_3) - \epsilon\gamma(p_1 + \tilde{p}_1), \\ \tilde{m}_3 - m_3 = 0, \\ \tilde{p}_1 - p_1 = \epsilon\alpha(p_2 \tilde{m}_3 + \tilde{p}_2 m_3) - \epsilon(p_3 \tilde{m}_2 + \tilde{p}_3 m_2), \\ \tilde{p}_2 - p_2 = \epsilon(p_3 \tilde{m}_1 + \tilde{p}_3 m_1) - \epsilon\alpha(p_1 \tilde{m}_3 + \tilde{p}_1 m_3), \\ \tilde{p}_3 - p_3 = \epsilon(p_1 \tilde{m}_2 + \tilde{p}_1 m_2 - p_2 \tilde{m}_1 - \tilde{p}_2 m_1). \end{cases}$$

As usual, this can be solved for (\tilde{m}, \tilde{p}) , thus yielding the reversible and birational map $x \mapsto \tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(1 + \epsilon B)x$, where $x = (m_1, m_2, m_3, p_1, p_2, p_3)^T$, and

$$A(x, \epsilon) = \begin{pmatrix} 1 & \epsilon(1 - \alpha)m_3 & \epsilon(1 - \alpha)m_2 & 0 & 0 & 0 \\ -\epsilon(1 - \alpha)m_3 & 1 & -\epsilon(1 - \alpha)m_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \epsilon p_3 & -\epsilon\alpha p_2 & 1 & -\epsilon\alpha m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon\alpha p_1 & \epsilon\alpha m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix} - \epsilon B,$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This map will be called dLT in the sequel. Obviously, m_3 serves as a conserved quantity for dLT. The remaining three conserved quantities can be found with the help of the HK bases approach. A simple conserved quantity can be found from the following statement which serves as a natural discretization of the Wronskian relation (68).

Proposition 41. *The set $\Gamma = (\tilde{m}_1 p_1 - m_1 \tilde{p}_1, \tilde{m}_2 p_2 - m_2 \tilde{p}_2, \tilde{m}_3 p_3 - m_3 \tilde{p}_3)$ is a HK basis for the map dLT with $\dim K_\Gamma(x) = 1$. At each point $x \in \mathbb{R}^6$ we have: $K_\Gamma(x) = [1 : 1 : b_3]$, where b_3 is a conserved quantity of dLT given by*

$$b_3 = \frac{(2\alpha - 1)m_3 + \epsilon^2(\alpha - 1)m_3(m_1^2 + m_2^2) + \epsilon^2\gamma(m_1 p_1 + m_2 p_2)}{m_3 \Delta_1}, \quad (69)$$

where

$$\Delta_1 = 1 + \epsilon^2\alpha(1 - \alpha)m_3^2 - \epsilon^2\gamma p_3. \quad (70)$$

Proof. A straightforward computation with MAPLE of the quantity

$$b_3 = -\frac{(\tilde{m}_1 p_1 - m_1 \tilde{p}_1) + (\tilde{m}_2 p_2 - m_2 \tilde{p}_2)}{(\tilde{m}_3 p_3 - m_3 \tilde{p}_3)}$$

leads to the value (69). It is an even function of ϵ and therefore a conserved quantity. \square

Further integrals of motion were found by Hirota and Kimura. We reproduce here their results with new simplified proofs.

Proposition 42. [12]

(a) *The set $\Phi = (m_1^2 + m_2^2, p_1 m_1 + p_2 m_2, p_1^2 + p_2^2, p_3^2, p_3, 1)$ is a HK basis for the map dLT with $\dim K_\Phi(x) = 3$.*

(b) *The set $\Phi_1 = (1, p_3, p_3^2, m_1^2 + m_2^2)$ is a HK basis for the map dLT with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$. The functions c_0, c_1, c_2 are conserved quantities of the map dLT , given by*

$$\begin{aligned} c_0 &= \frac{m_1^2 + m_2^2 + 2\gamma p_3 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{\Delta_1 \Delta_2}, \\ c_1 &= -\frac{2\gamma(1 - \epsilon^2 \alpha(1 - \alpha)m_3^2)(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)})}{\Delta_1 \Delta_2}, \\ c_2 &= -\frac{\epsilon^2 \gamma^2(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)})}{\Delta_1 \Delta_2}. \end{aligned}$$

Here Δ_1 is given in (70), and $\Delta_2 = 1 + \epsilon^2 \Delta_2^{(2)} + \epsilon^4 \Delta_2^{(4)} + \epsilon^6 \Delta_2^{(6)}$; coefficients $\Delta^{(q)}$ and $c_k^{(q)}$ are polynomials of degree q in the phase variables. In particular:

$$\begin{aligned} c_2^{(2)} &= m_1^2 + m_2^2 + (1 - 2\alpha + 2\alpha^2)m_3^2 - 2\gamma p_3, \\ \Delta_2^{(2)} &= m_1^2 + m_2^2 + (1 - 3\alpha + 3\alpha^2)m_3^2 - \gamma p_3. \end{aligned}$$

(c) *The set $\Phi_2 = (1, p_3, p_3^2, m_1 p_1 + m_2 p_2)$ is a HK basis for the map dLT with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_2}(x) = [d_0 : d_1 : d_2 : -1]$. The functions d_0, d_1, d_2 are conserved quantities of the map dLT , given by*

$$\begin{aligned} d_0 &= \frac{m_1 p_1 + m_2 p_2 + m_3 p_3 + \epsilon^2 d_0^{(4)} + \epsilon^4 d_0^{(6)} + \epsilon^6 d_0^{(8)} + \epsilon^8 d_0^{(10)}}{\Delta_1 \Delta_2}, \\ d_1 &= -\frac{m_3 + \epsilon^2 d_1^{(3)} + \epsilon^4 d_1^{(5)} + \epsilon^6 d_1^{(7)} + \epsilon^8 d_1^{(9)}}{\Delta_1 \Delta_2}, \\ d_2 &= -\frac{\epsilon^2 \gamma(1 - \alpha)m_3(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)})}{\Delta_1 \Delta_2}, \end{aligned}$$

where $d_k^{(q)}$ are polynomials of degree q in the phase variables. In particular,

$$d_1^{(3)} = \gamma(m_1 p_1 + m_2 p_2) - \gamma(3 - 2\alpha)m_3 p_3 + \alpha m_3(m_1^2 + m_2^2) + (1 - 3\alpha + 3\alpha^2)m_3^3.$$

(d) *The set $\Phi_3 = (1, p_3, p_3^2, p_1^2 + p_2^2)$ is a HK basis for the map dLT with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_3}(x) = [e_0 : e_1 : e_2 : -1]$. The functions e_0, e_1, e_2 are conserved quantities of the map dLT , given by*

$$\begin{aligned} e_0 &= \frac{p_1^2 + p_2^2 + p_3^2 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1 \Delta_2}, \\ e_1 &= -\frac{2\epsilon^2(e_1^{(3)} + \epsilon^2 e_1^{(5)} + \epsilon^4 e_1^{(7)} + \epsilon^6 e_1^{(9)})}{\Delta_1 \Delta_2}, \\ e_2 &= -\frac{(1 + \epsilon^2(1 - \alpha)^2 m_3^2)(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)})}{\Delta_1 \Delta_2}, \end{aligned}$$

where $e_k^{(q)}$ are polynomials of degree q in the phase variables. In particular,

$$e_1^{(3)} = \gamma(p_1^2 + p_2^2 + p_3^2) - (1 - \alpha)m_3(m_1p_1 + m_2p_2 + m_3p_3).$$

Proof. (b) We consider a linear system of equations

$$(c_0 + c_1p_3 + c_2p_3^2) \circ f^i(m, p, \epsilon) = (m_1^2 + m_2^2) \circ f^i(m, p, \epsilon), \quad (71)$$

for all $i \in \mathbb{Z}$. Numerically one sees that it admits a unique solution, and one can identify the linear relation

$$\frac{1}{2}\gamma\epsilon^2c_1 = (1 - \epsilon^2\alpha(1 - \alpha)m_3^2)c_2. \quad (72)$$

The system of three equations for three unknowns c_0, c_1, c_2 consisting of (71) with $i = 0, 1$ and (72) can easily be solved with MAPLE. Its solutions are even functions of ϵ , which proves that they are integrals of motion.

(c) This time we consider the linear system of equations

$$(d_0 + d_1p_3 + d_2p_3^2) \circ f^i(m, p, \epsilon) = (m_1p_1 + m_2p_2) \circ f^i(m, p, \epsilon), \quad (73)$$

for all $i \in \mathbb{Z}$. Numerically we see that it admits a unique solution, and we can identify the linear relation

$$\gamma d_2 = (1 - \alpha)m_3c_2. \quad (74)$$

The system of three equations for the three unknowns d_0, d_1, d_2 consisting of (73) for $i = 0, 1$ and of (74) with c_2 already found in part b) can easily be solved with MAPLE. Its solutions are even functions of ϵ and therefore are integrals.

(d) Completely analogous to the last two proofs: we solve the linear system of three equations for the three unknowns e_0, e_1, e_2 , consisting of the equations

$$(e_0 + e_1p_3 + e_2p_3^2) \circ f^i(m, p, \epsilon) = (p_1^2 + p_2^2) \circ f^i(m, p, \epsilon),$$

for $i = 0, 1$, and of the linear relation

$$\epsilon^2\gamma^2e_2 = (1 + \epsilon^2(1 - \alpha)^2m_3^2)c_2,$$

and verify that they are even functions of ϵ . □

We note that for $\alpha = 1$ the integrals d_0, d_1, d_2 simplify to

$$d_0 = \frac{m_1p_1 + m_2p_2 + m_3p_3}{1 - \epsilon^2\gamma p_3}, \quad d_1 = -\frac{m_3 + \epsilon^2\gamma(m_1p_1 + m_2p_2)}{1 - \epsilon^2\gamma p_3}, \quad d_2 = 0.$$

It is possible to find a further simple, in fact polynomial, integral for the map dLT.

Proposition 43. [12] *The function*

$$F = m_1^2 + m_2^2 + 2\gamma p_3 - \epsilon^2((1 - \alpha)m_3m_1 + \gamma p_1)^2 - \epsilon^2((1 - \alpha)m_3m_2 + \gamma p_2)^2,$$

is a conserved quantity for the map dLT.

Proof. Setting $C = 1 - \epsilon^2(1 - \alpha)^2m_3^2$, $D = -2\epsilon^2\gamma(1 - \alpha)m_3$, $E = -\epsilon^2\gamma^2$, one can check that $Cc_1 + Dd_1 + Ee_1 = 0$ and $Cc_2 + Dd_2 + Ee_2 = -2\gamma$. This yields for the conserved quantity $F = Cc_0 + Dd_0 + Ee_0$ the expression given in the Proposition. □

Considering the leading terms of the power expansions in ϵ , one sees immediately that the integrals c_0 , d_0 , e_0 , and m_3 are functionally independent. Using exact evaluation of gradients we can also verify independence of other sets of integrals. It turns out that for $\alpha \neq 1$ each one of the quadruples $\{d_0, d_1, d_2, m_3\}$ and $\{e_0, e_1, e_2, m_3\}$ consists of independent integrals.

A direct “bilinearization” of the HK bases of Proposition 42 provides us with an alternative source of integrals of motion:

Proposition 44. *The set*

$$\Psi = (m_1\tilde{m}_1 + m_2\tilde{m}_2, p_1\tilde{m}_1 + \tilde{p}_1m_1 + p_2\tilde{m}_2 + \tilde{p}_2m_2, p_1\tilde{p}_1 + p_2\tilde{p}_2, p_3\tilde{p}_3, p_3 + \tilde{p}_3, 1)$$

is a HK basis for the map dLT with $\dim K_\Psi(x) = 3$. Each of the following subsets of Ψ ,

$$\Psi_1 = (1, p_3 + \tilde{p}_3, p_3\tilde{p}_3, m_1\tilde{m}_1 + m_2\tilde{m}_2),$$

$$\Psi_2 = (1, p_3 + \tilde{p}_3, p_3\tilde{p}_3, m_1\tilde{p}_1 + \tilde{m}_1p_1 + m_2\tilde{p}_2 + \tilde{m}_2p_2),$$

$$\Psi_3 = (1, p_3 + \tilde{p}_3, p_3\tilde{p}_3, p_1\tilde{p}_1 + p_2\tilde{p}_2),$$

is a HK basis with a one-dimensional null-space.

Concerning solutions of dLT as functions of the (discrete) time t , the crucial result is given in the following statement which should be considered as the proper discretization of eq. (67).

Proposition 45. [12] *The component p_3 of the solution of difference equations (69) satisfies a relation of the type*

$$Q(p_3, \tilde{p}_3) = q_0p_3^2\tilde{p}_3^2 + q_1p_3\tilde{p}_3(p_3 + \tilde{p}_3) + q_2(p_3^2 + \tilde{p}_3^2) + q_3p_3\tilde{p}_3 + q_4(p_3 + \tilde{p}_3) + q_5 = 0,$$

coefficients of the biquadratic polynomial Q being conserved quantities of dLT. Hence, $p_3(t)$ is an elliptic function of degree 2.

Although it remains unknown whether the map dLT admits an invariant Poisson structure, we have the following statement.

Proposition 46. *The map dLT possesses an invariant volume form:*

$$\det \frac{\partial \tilde{x}}{\partial x} = \frac{\phi(\tilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(x)},$$

with $\phi(x) = \Delta_2(x, \epsilon)$.

13. KIRCHHOFF CASE OF THE RIGID BODY MOTION IN AN IDEAL FLUID

The motion of a rigid body in an ideal fluid is described by Kirchhoff equations (64) with H being a quadratic form in $m = (m_1, m_2, m_3)^T \in \mathbb{R}^3$ and $p = (p_1, p_2, p_3)^T \in \mathbb{R}^3$. The physical meaning of m is the total angular momentum, whereas p represents the total linear momentum of the system. A detailed introduction to the general context of rigid body dynamics and its mathematical foundations can be found in [14].

The integrable case of this system found in the original paper by Kirchhoff [13] and carrying his name is characterized by the Hamilton function $H = H_1/2$, where

$$H_1 = a_1(m_1^2 + m_2^2) + a_3m_3^2 + b_1(p_1^2 + p_2^2) + b_3p_3^2.$$

The differential equations of the Kirchhoff case are:

$$\begin{cases} \dot{m}_1 = (a_3 - a_1)m_2m_3 + (b_3 - b_1)p_2p_3, \\ \dot{m}_2 = (a_1 - a_3)m_1m_3 + (b_1 - b_3)p_1p_3, \\ \dot{m}_3 = 0, \\ \dot{p}_1 = a_3p_2m_3 - a_1p_3m_2, \\ \dot{p}_2 = a_1p_3m_1 - a_3p_1m_3, \\ \dot{p}_3 = a_1(p_1m_2 - p_2m_1). \end{cases} \quad (75)$$

Along with the Hamilton function H and the Casimir functions (65), it possesses the obvious fourth integral, due to the rotational symmetry of the system: $H_2 = m_3$. Traditionally, the explicit integration of the Kirchhoff case in terms of elliptic functions starts with the following observation: the component p_3 of the solution satisfies the differential equation

$$\dot{p}_3^2 = P_4(p_3),$$

with a quartic polynomial P_4 whose coefficients are expressed through integrals of motion:

$$P_4(p_3) = a_1(H_1 - a_3m_3^2 - b_1(C_1 - p_3^2) - b_3p_3^2)(C_1 - p_3^2) - a_1^2(C_2 - m_3p_3)^2.$$

We mention also the following Wronskian relation which follows easily from equations of motion:

$$a_1(\dot{m}_1p_1 - m_1\dot{p}_1) + a_1(\dot{m}_2p_2 - m_2\dot{p}_2) + (2a_3 - a_1)(\dot{m}_3p_3 - m_3\dot{p}_3) = 0. \quad (76)$$

Applying the HK approach to (75), we obtain the following system of equations:

$$\begin{cases} \tilde{m}_1 - m_1 = \epsilon(a_3 - a_1)(\tilde{m}_2m_3 + m_2\tilde{m}_3) + \epsilon(b_3 - b_1)(\tilde{p}_2p_3 + p_2\tilde{p}_3), \\ \tilde{m}_2 - m_2 = \epsilon(a_1 - a_3)(\tilde{m}_1m_3 + m_1\tilde{m}_3) + \epsilon(b_1 - b_3)(\tilde{p}_1p_3 + p_1\tilde{p}_3), \\ \tilde{m}_3 - m_3 = 0, \\ \tilde{p}_1 - p_1 = \epsilon a_3(\tilde{p}_2m_3 + p_2\tilde{m}_3) - \epsilon a_1(\tilde{p}_3m_2 + p_3\tilde{m}_2), \\ \tilde{p}_2 - p_2 = \epsilon a_1(\tilde{p}_3m_1 + p_3\tilde{m}_1) - \epsilon a_3(\tilde{p}_1m_3 + p_1\tilde{m}_3), \\ \tilde{p}_3 - p_3 = \epsilon a_1(\tilde{p}_1m_2 + p_1\tilde{m}_2) - \epsilon a_1(\tilde{p}_2m_1 + p_2\tilde{m}_1). \end{cases}$$

As usual, these equations define a birational map $\tilde{x} = f(x, \epsilon)$, $x = (m, p)^T$. We will refer to this map as dK. Like in the case of dLT, m_3 is a conserved quantity of dK. A further “simple” conserved quantity can be found from the following natural discretization of the Wronskian relation (76).

Proposition 47. *The set $\Gamma = (\tilde{m}_1p_1 - m_1\tilde{p}_1, \tilde{m}_2p_2 - m_2\tilde{p}_2, \tilde{m}_3p_3 - m_3\tilde{p}_3)$ is a HK basis for the map dK with $\dim K_\Gamma(x) = 1$. At each point $x \in \mathbb{R}^6$ we have: $K_\Gamma(x) = [1 : 1 : -\gamma_3]$, where γ_3 is a conserved quantity of dK given by*

$$\gamma_3 = \frac{\Delta_0}{a_1\Delta_1}, \quad (77)$$

where

$$\Delta_0 = a_1 - 2a_3 + \epsilon^2 a_1^2(a_1 - a_3)(m_1^2 + m_2^2) + \epsilon^2 a_1 a_3(b_1 - b_3)(p_1^2 + p_2^2), \quad (78)$$

$$\Delta_1 = 1 + \epsilon^2 a_3(a_1 - a_3)m_3^2 + \epsilon^2 a_1(b_1 - b_3)p_3^2. \quad (79)$$

Proof. Like in the case of dLT, we let MAPLE compute the quantity

$$\gamma_3 = \frac{(\tilde{m}_1 p_1 - m_1 \tilde{p}_1) + (\tilde{m}_2 p_2 - m_2 \tilde{p}_2)}{(\tilde{m}_3 p_3 - m_3 \tilde{p}_3)},$$

which results in (77), an even function of ϵ and therefore a conserved quantity. \square

Interestingly enough, this same integral may also be obtained from another HK basis:

Proposition 48. *The set $\Phi_0 = (m_1^2 + m_2^2, p_1^2 + p_2^2, p_3^2, 1)$ is a HK Basis for the map dK with $\dim K_{\Phi_0}(x) = 1$. The linear combination of these functions vanishing along the orbits can be put as $\Delta_0 - \gamma_3 a_1 \Delta_1 = 0$.*

Proof. The statement of the Proposition deals with the solution of a linear system of equations consisting of

$$(c_1(m_1^2 + m_2^2) + c_2(p_1^2 + p_2^2) + c_3 p_3^2) \circ f^i(m, p, \epsilon) = 1 \quad (80)$$

for all $i \in \mathbb{Z}$. We solve this system with $i = -1, 0, 1$ (numerically or symbolically), and observe that the solutions satisfy $a_3(b_1 - b_3)c_1 = a_1(a_1 - a_3)c_2$. Then, we consider the system of three equations for c_1, c_2, c_3 consisting of the latter linear relation between c_1, c_2 , and of eqs. (80) for $i = 0, 1$. This system is easily solved symbolically (by MAPLE), its unique solution can be put as in the Proposition. Its components are manifestly even functions of ϵ , thus conserved quantities. \square

Proposition 49.

(a) *The set $\Phi = (m_1^2 + m_2^2, p_1 m_1 + p_2 m_2, p_1^2 + p_2^2, p_3^2, p_3, 1)$ is a HK basis for the map dK with $\dim K_{\Phi}(x) = 3$.*

(b) *The set $\Phi_1 = (1, p_3, p_3^2, m_1^2 + m_2^2)$ is a HK basis for the map dK with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$. The functions c_0, c_1, c_2 are conserved quantities of the map dK , given by*

$$\begin{aligned} c_0 &= \frac{a_1(m_1^2 + m_2^2) - (b_1 - b_3)p_3^2 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{a_1 \Delta_1 \Delta_2}, \\ c_1 &= -\frac{2\epsilon^2 a_3(b_1 - b_3)m_3(C_2 + \epsilon^2 c_1^{(4)} + \epsilon^4 c_1^{(6)} + \epsilon^6 c_1^{(8)})}{\Delta_1 \Delta_2}, \\ c_2 &= \frac{(b_1 - b_3)(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)} + \epsilon^8 c_2^{(8)})}{a_1 \Delta_1 \Delta_2}, \end{aligned}$$

where Δ_1 is given in (79), and $\Delta_2 = 1 + \epsilon^2 \Delta_2^{(2)} + \epsilon^4 \Delta_2^{(4)} + \epsilon^6 \Delta_2^{(6)}$; coefficients $c_k^{(q)}$ and $\Delta_2^{(q)}$ are homogeneous polynomials of degree q in the phase variables. In particular:

$$\begin{aligned} c_2^{(2)} &= -2a_1^2(m_1^2 + m_2^2) - (a_1^2 - 2a_1 a_3 + 3a_3^2)m_3^2 + a_1(b_1 - b_3)(p_1^2 + p_2^2) - a_1(b_1 - b_3)p_3^2, \\ \Delta_2^{(2)} &= a_1^2(m_1^2 + m_2^2) + (a_1^2 - 3a_1 a_3 + 3a_3^2)m_3^2 - a_1(b_1 - b_3)(p_1^2 + p_2^2) + a_1(b_1 - b_3)p_3^2. \end{aligned}$$

(c) *The set $\Phi_2 = (1, p_3, p_3^2, m_1 p_1 + m_2 p_2)$ is a HK basis for the map dK with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_2}(x) = [d_0 : d_1 : d_2 : -1]$. The*

functions d_0, d_1, d_2 are conserved quantities of the map dK , given by

$$\begin{aligned} d_0 &= \frac{C_2 + \epsilon^2 d_0^{(4)} + \epsilon^4 d_0^{(6)} + \epsilon^6 d_0^{(8)} + \epsilon^8 d_0^{(10)}}{\Delta_1 \Delta_2}, \\ d_1 &= \frac{m_3(-1 + \epsilon^2 d_1^{(2)} + \epsilon^4 d_1^{(4)} + \epsilon^6 d_1^{(6)} + \epsilon^8 d_1^{(8)})}{\Delta_1 \Delta_2}, \\ d_2 &= \frac{a_1(b_3 - b_1)\epsilon^2(C_2 + \epsilon^2 c_1^{(4)} + \epsilon^4 c_1^{(6)} + \epsilon^6 c_1^{(8)})}{\Delta_1 \Delta_2}, \end{aligned}$$

where $d_k^{(q)}$ are homogeneous polynomials of degree q in the phase variables. In particular, $d_1^{(2)} = -a_1 a_3(m_1^2 + m_2^2) - (a_1^2 - 3a_1 a_3 + 3a_3^2)m_3^2 + (a_1 - a_3)(b_1 - b_3)(p_1^2 + p_2^2) - 3a_1(b_1 - b_3)p_3^2$.

(d) The set $\Phi_3 = (1, p_3, p_3^2, p_1^2 + p_2^2)$ is a HK basis for the map dK with a one-dimensional null-space. At each point $x \in \mathbb{R}^6$ we have: $K_{\Phi_3}(x) = [e_0 : e_1 : e_2 : -1]$. The functions e_0, e_1, e_2 are conserved quantities of the map dK , given by

$$\begin{aligned} e_0 &= \frac{C_1 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1 \Delta_2}, \\ e_1 &= \frac{2\epsilon^2 a_1(a_3 - a_1)m_3(C_2 + \epsilon^2 c_1^{(4)} + \epsilon^4 c_1^{(6)} + \epsilon^6 c_1^{(8)})}{\Delta_1 \Delta_2}, \\ e_2 &= \frac{-1 + \epsilon^2 e_2^{(2)} + \epsilon^4 e_2^{(4)} + \epsilon^6 e_2^{(6)} + \epsilon^8 e_2^{(8)}}{\Delta_1 \Delta_2}, \end{aligned}$$

where $e_k^{(q)}$ are polynomials of degree q in the phase variables. In particular,

$$e_2^{(2)} = -a_1^2(m_1^2 + m_2^2) - (2a_1^2 - 4a_1 a_3 + 3a_3^2)m_3^2 + 2a_1(b_1 - b_3)(p_1^2 + p_2^2) - a_1(b_1 - b_3)p_3^2.$$

Proof. Statement (b) is proven using direct calculation. Statements (c) and (d) then follow analogously to Proposition 42 from the existence of linear relations between c_1 and d_2 , as well as between c_1 and e_1 . \square

One can show that each of the sets $\{c_0, c_1, c_2\}$, $\{d_0, d_1, d_2\}$, and $\{e_0, e_1, e_2\}$ consists of three independent integrals of motion. Moreover, each of the sets $\{c_0, c_1, c_2, m_3\}$ and $\{e_0, e_1, e_2, m_3\}$ consists of four independent integrals. As further important results, we mention that Propositions 44 (on the “bilinear” HK bases), 45 (on the invariant bi-quadratic curve for (p_3, \tilde{p}_3)), and 46 (on the invariant measure) hold literally true for the map dK .

14. CLEBSCH CASE OF THE RIGID BODY MOTION IN AN IDEAL FLUID

Another famous integrable case of the Kirchhoff equations was discovered by Clebsch [4] and is characterized by the Hamilton function $H = H_1/2$, where

$$H_1 = \langle m, Am \rangle + \langle p, Bp \rangle = \frac{1}{2} \sum_{k=1}^3 (a_k m_k^2 + b_k p_k^2),$$

where $A = \text{diag}(a_1, a_2, a_3)$ and $B = \text{diag}(b_1, b_2, b_3)$ satisfy the condition

$$\frac{b_1 - b_2}{a_3} + \frac{b_2 - b_3}{a_1} + \frac{b_3 - b_1}{a_2} = 0. \quad (81)$$

This condition is also equivalent to saying that the quantity

$$\theta = \frac{b_j - b_k}{a_i(a_j - a_k)} \quad (82)$$

takes one and the same value for all permutations (i, j, k) of the indices $(1, 2, 3)$.

For an embedding of this system into the modern theory of integrable systems see [16, 22]. Note that the Kirchhoff case ($a_1 = a_2$ and $b_1 = b_2$) can be considered as a particular case of the Clebsch case, but is special in many respects (the symmetry resulting in the existence of the Noether integral m_3 , solvability in elliptic functions, in contrast to the general Clebsch system being solvable in terms of theta-functions of genus 2, etc.). Equations of motion of the Clebsch case are:

$$\begin{cases} \dot{m} = m \times Am + p \times Bp, \\ \dot{p} = p \times Am, \end{cases} \quad (83)$$

or in components

$$\begin{cases} \dot{m}_1 = (a_3 - a_2)m_2m_3 + (b_3 - b_2)p_2p_3, \\ \dot{m}_2 = (a_1 - a_3)m_3m_1 + (b_1 - b_3)p_3p_1, \\ \dot{m}_3 = (a_2 - a_1)m_1m_2 + (b_2 - b_1)p_1p_2, \\ \dot{p}_1 = a_3m_3p_2 - a_2m_2p_3, \\ \dot{p}_2 = a_1m_1p_3 - a_3m_3p_1, \\ \dot{p}_3 = a_2m_2p_1 - a_1m_1p_2. \end{cases} \quad (84)$$

Condition (81) can be resolved for a_i as

$$a_1 = \frac{b_2 - b_3}{\omega_2 - \omega_3}, \quad a_2 = \frac{b_3 - b_1}{\omega_3 - \omega_1}, \quad a_3 = \frac{b_1 - b_2}{\omega_1 - \omega_2}.$$

For fixed values of ω_i and varying values of b_i , equations of motion of the Clebsch case share the integrals of motion: the Casimirs C_1, C_2 , cf. eq. (65), and the Hamiltonians

$$I_i = p_i^2 + \frac{m_j^2}{\omega_i - \omega_k} + \frac{m_k^2}{\omega_i - \omega_j}.$$

There are four independent functions among C_i, I_i , because of $C_1 = I_1 + I_2 + I_3$. Note that $H_1 = b_1I_1 + b_2I_2 + b_3I_3$. One can denote all models with the same ω_i as a hierarchy, single flows of which are characterized by the parameters b_i . Usually, one denotes as “the first flow” of this hierarchy the one corresponding to the choice $b_i = \omega_i$, so that $a_i = 1$. Thus, the first flow is characterized by the value $\theta = \infty$ of the constant (82).

14.1. First flow of the Clebsch system. The first flow of the Clebsch hierarchy is generated by the Hamilton function $H = H_1/2$, where

$$H_1 = m_1^2 + m_2^2 + m_3^2 + \omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2.$$

The corresponding equations of motion read:

$$\begin{cases} \dot{m} = p \times \Omega p, \\ \dot{p} = p \times m, \end{cases}$$

where $\Omega = \text{diag}(\omega_1, \omega_2, \omega_3)$ is the matrix of parameters, or in components:

$$\begin{cases} \dot{m}_1 = (\omega_3 - \omega_2)p_2p_3, \\ \dot{m}_2 = (\omega_1 - \omega_3)p_3p_1, \\ \dot{m}_3 = (\omega_2 - \omega_1)p_1p_2, \\ \dot{p}_1 = m_3p_2 - m_2p_3, \\ \dot{p}_2 = m_1p_3 - m_3p_1, \\ \dot{p}_3 = m_2p_1 - m_1p_2. \end{cases}$$

The fourth independent quadratic integral can be chosen as

$$H_2 = \omega_1 m_1^2 + \omega_2 m_2^2 + \omega_3 m_3^2 - \omega_2 \omega_3 p_1^2 - \omega_3 \omega_1 p_2^2 - \omega_1 \omega_2 p_3^2.$$

Note that $H_1 = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3$, $H_1 = -\omega_2 \omega_3 I_1 - \omega_3 \omega_1 I_2 - \omega_1 \omega_2 I_3$.

We mention the following Wronskian relation:

$$(\dot{m}_1 p_1 - m_1 \dot{p}_1) + (\dot{m}_2 p_2 - m_2 \dot{p}_2) + (\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0, \quad (85)$$

which holds true for the first Clebsch flow.

The HK discretization of the first Clebsch flow (proposed in [21]) is:

$$\begin{cases} \tilde{m}_1 - m_1 = \epsilon(\omega_3 - \omega_2)(\tilde{p}_2 p_3 + p_2 \tilde{p}_3), \\ \tilde{m}_2 - m_2 = \epsilon(\omega_1 - \omega_3)(\tilde{p}_3 p_1 + p_3 \tilde{p}_1), \\ \tilde{m}_3 - m_3 = \epsilon(\omega_2 - \omega_1)(\tilde{p}_1 p_2 + p_1 \tilde{p}_2), \\ \tilde{p}_1 - p_1 = \epsilon(\tilde{m}_3 p_2 + m_3 \tilde{p}_2) - \epsilon(\tilde{m}_2 p_3 + m_2 \tilde{p}_3), \\ \tilde{p}_2 - p_2 = \epsilon(\tilde{m}_1 p_3 + m_1 \tilde{p}_3) - \epsilon(\tilde{m}_3 p_1 + m_3 \tilde{p}_1), \\ \tilde{p}_3 - p_3 = \epsilon(\tilde{m}_2 p_1 + m_2 \tilde{p}_1) - \epsilon(\tilde{m}_1 p_2 + m_1 \tilde{p}_2). \end{cases}$$

As usual, it leads to a reversible birational map $\tilde{x} = f(x, \epsilon)$, $x = (m, p)^T$, given by $f(x, \epsilon) = A^{-1}(x, \epsilon)x$ with

$$A(m, p, \epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon\omega_{23}p_3 & \epsilon\omega_{23}p_2 \\ 0 & 1 & 0 & \epsilon\omega_{31}p_3 & 0 & \epsilon\omega_{31}p_1 \\ 0 & 0 & 1 & \epsilon\omega_{12}p_2 & \epsilon\omega_{12}p_1 & 0 \\ 0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix},$$

where the abbreviation $\omega_{ij} = \omega_i - \omega_j$ is used. This map will be referred to as dC.

A “simple” conserved quantity can be found from the following natural discretization of the Wronskian relation (85).

Proposition 50. *The set $\Gamma = (\tilde{m}_1 p_1 - m_1 \tilde{p}_1, \tilde{m}_2 p_2 - m_2 \tilde{p}_2, \tilde{m}_3 p_3 - m_3 \tilde{p}_3)$ is a HK basis for the map dC with $\dim K_\Gamma(x) = 1$. At each point $x \in \mathbb{R}^6$ we have: $K_\Gamma(x) = [e_1 : e_2 : e_3]$, where*

$$e_i = 1 + \epsilon^2(\omega_i - \omega_j)p_j^2 + \epsilon^2(\omega_i - \omega_k)p_k^2. \quad (86)$$

The conserved quantities e_i/e_j can be put as $e_i/e_j = (1 + \epsilon^2\omega_i J)/(1 + \epsilon^2\omega_j J)$, where J is a nice and symmetric integral,

$$J = \frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)}.$$

Remarkably, it can be obtained also from a different (monomial) HK basis, see part b) of the following statement.

Proposition 51. [17]

(a) *The set of functions $\Phi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1p_1, m_2p_2, m_3p_3, 1)$ is a HK basis for the map dC with $\dim K_\Phi(m, p) = 4$. Thus, any orbit of the map dC lies on an intersection of four quadrics in \mathbb{R}^6 .*

(b) *The set of functions $\Phi_0 = (p_1^2, p_2^2, p_3^2, 1)$ is a HK basis for the map dC with $\dim K_{\Phi_0}(m, p) = 1$. At each point $(m, p) \in \mathbb{R}^6$ there holds:*

$$\begin{aligned} K_{\Phi_0}(m, p) &= [e_1 : e_2 : e_3 : -(p_1^2 + p_2^2 + p_3^2)] \\ &= \left[\frac{1}{J} + \epsilon^2 \omega_1 : \frac{1}{J} + \epsilon^2 \omega_2 : \frac{1}{J} + \epsilon^2 \omega_3 : -1 \right], \end{aligned}$$

with the quantities e_i given in (86).

(c) *The sets of functions*

$$\Phi_1 = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1p_1), \quad (87)$$

$$\Phi_2 = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_2p_2), \quad (88)$$

$$\Phi_3 = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_3p_3), \quad (89)$$

are HK bases for the map dC with $\dim K_{\Phi_1}(m, p) = \dim K_{\Phi_2}(m, p) = \dim K_{\Phi_3}(m, p) = 1$. At each point $(m, p) \in \mathbb{R}^6$ there holds:

$$K_{\Phi_1}(m, p) = [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5 : \alpha_6 : -1],$$

$$K_{\Phi_2}(m, p) = [\beta_1 : \beta_2 : \beta_3 : \beta_4 : \beta_5 : \beta_6 : -1],$$

$$K_{\Phi_3}(m, p) = [\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : \gamma_5 : \gamma_6 : -1],$$

where α_j, β_j , and γ_j are rational functions of (m, p) , even with respect to ϵ . They are conserved quantities of the map dCS . For $j = 1, 2, 3$, they are of the form

$$h = \frac{h^{(2)} + \epsilon^2 h^{(4)} + \epsilon^4 h^{(6)} + \epsilon^6 h^{(8)} + \epsilon^8 h^{(10)} + \epsilon^{10} h^{(12)}}{2\epsilon^2(p_1^2 + p_2^2 + p_3^2)\Delta},$$

where h stands for any of the functions $\alpha_j, \beta_j, \gamma_j$, $j = 1, 2, 3$,

$$\Delta = m_1p_1 + m_2p_2 + m_3p_3 + \epsilon^2 \Delta^{(4)} + \epsilon^4 \Delta^{(6)} + \epsilon^6 \Delta^{(8)},$$

and the corresponding $h^{(2q)}, \Delta^{(2q)}$ are homogeneous polynomials in phase variables of degree $2q$. For instance,

$$\begin{aligned} \alpha_1^{(2)} &= C_1 - I_1, & \alpha_2^{(2)} &= -I_1, & \alpha_3^{(2)} &= -I_1, \\ \beta_1^{(2)} &= -I_2, & \beta_2^{(2)} &= C_1 - I_2, & \beta_3^{(2)} &= -I_2, \\ \gamma_1^{(2)} &= -I_3, & \gamma_2^{(2)} &= -I_3, & \gamma_3^{(2)} &= C_1 - I_3. \end{aligned}$$

For $j = 4, 5, 6$, the functions $\alpha_j, \beta_j, \gamma_j$ are given by

$$\begin{pmatrix} \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_4 & \beta_5 & \beta_6 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} = \begin{pmatrix} D & A_1/(\omega_1 - \omega_3) & A_1/(\omega_1 - \omega_2) \\ A_2/(\omega_2 - \omega_3) & D & A_2/(\omega_2 - \omega_1) \\ A_3/(\omega_3 - \omega_2) & A_3/(\omega_3 - \omega_1) & D \end{pmatrix},$$

where

$$A_k = \frac{1 + \epsilon^2 A_k^{(2)} + \epsilon^4 A_k^{(4)} + \epsilon^6 A_k^{(6)} + \epsilon^8 A_k^{(8)}}{2\epsilon^2 \Delta},$$

$$D = \frac{p_1^2 + p_2^2 + p_3^2 + \epsilon^2 D^{(4)} + \epsilon^4 D^{(6)} + \epsilon^6 D^{(8)}}{2\Delta},$$

and $A_k^{(2q)}$, $D^{(2q)}$ are homogeneous polynomials of degree $2q$ in phase variables, for instance,

$$A_k^{(2)} = m_1^2 + m_2^2 + m_3^2 + (\omega_2 + \omega_3 - 2\omega_1)p_1^2 + (\omega_3 + \omega_1 - 2\omega_2)p_2^2 + (\omega_1 + \omega_2 - 2\omega_3)p_3^2.$$

The four conserved quantities J , α_1 , β_1 and γ_1 are functionally independent.

Our paper [17] contains a much more detailed information about the HK bases of the map dC, for instance, the further basis with a one-dimensional null-space: $\Theta = (p_1^2, p_2^2, p_3^2, m_1 p_1, m_2 p_2, m_3 p_3)$. However, the following finding about the “bilinear” versions of the above bases is new.

Proposition 52. *Each of the sets of functions*

$$\Psi_0 = (\tilde{p}_1 p_1, \tilde{p}_2 p_2, \tilde{p}_3 p_3, 1), \quad (90)$$

$$\Psi_1 = (\tilde{p}_1 p_1, \tilde{p}_2 p_2, \tilde{p}_3 p_3, \tilde{m}_1 m_1, \tilde{m}_2 m_2, \tilde{m}_3 m_3, \tilde{m}_1 p_1 + m_1 \tilde{p}_1), \quad (91)$$

$$\Psi_2 = (\tilde{p}_1 p_1, \tilde{p}_2 p_2, \tilde{p}_3 p_3, \tilde{m}_1 m_1, \tilde{m}_2 m_2, \tilde{m}_3 m_3, \tilde{m}_2 p_2 + m_2 \tilde{p}_2), \quad (92)$$

$$\Psi_3 = (\tilde{p}_1 p_1, \tilde{p}_2 p_2, \tilde{p}_3 p_3, \tilde{m}_1 m_1, \tilde{m}_2 m_2, \tilde{m}_3 m_3, \tilde{m}_3 p_3 + m_3 \tilde{p}_3), \quad (93)$$

is a HK basis for the map dC with a one-dimensional null-space.

14.2. General flow of the Clebsch system. The HK discretization of the flow (83) reads

$$\begin{cases} \tilde{m} - m = \epsilon(\tilde{m} \times Am + m \times A\tilde{m} + \tilde{p} \times Bp + p \times B\tilde{p}), \\ \tilde{p} - p = \epsilon(\tilde{p} \times Am + p \times A\tilde{m}), \end{cases}$$

in components:

$$\begin{cases} \tilde{m}_1 - m_1 = \epsilon(a_3 - a_2)(\tilde{m}_2 m_3 + m_2 \tilde{m}_3) + \epsilon(b_3 - b_2)(\tilde{p}_2 p_3 + p_2 \tilde{p}_3), \\ \tilde{m}_2 - m_2 = \epsilon(a_1 - a_3)(\tilde{m}_3 m_1 + m_3 \tilde{m}_1) + \epsilon(b_1 - b_3)(\tilde{p}_3 p_1 + p_3 \tilde{p}_1), \\ \tilde{m}_3 - m_3 = \epsilon(a_2 - a_1)(\tilde{m}_1 m_2 + m_1 \tilde{m}_2) + \epsilon(b_2 - b_1)(\tilde{p}_1 p_2 + p_1 \tilde{p}_2), \\ \tilde{p}_1 - p_1 = \epsilon a_3(\tilde{m}_3 p_2 + m_3 \tilde{p}_2) - \epsilon a_2(\tilde{m}_2 p_3 + m_2 \tilde{p}_3), \\ \tilde{p}_2 - p_2 = \epsilon a_1(\tilde{m}_1 p_3 + m_1 \tilde{p}_3) - \epsilon a_3(\tilde{m}_3 p_1 + m_3 \tilde{p}_1), \\ \tilde{p}_3 - p_3 = \epsilon a_2(\tilde{m}_2 p_1 + m_2 \tilde{p}_1) - \epsilon a_1(\tilde{m}_1 p_2 + m_1 \tilde{p}_2). \end{cases} \quad (94)$$

In what follows, we will use the abbreviations $b_{ij} = b_i - b_j$ and $a_{ij} = a_i - a_j$. The linear system (94) defines an explicit, birational map $f : \mathbb{R}^6 \rightarrow \mathbb{R}^6$,

$$\begin{pmatrix} \tilde{m} \\ \tilde{p} \end{pmatrix} = f(m, p, \epsilon) = M^{-1}(m, p, \epsilon) \begin{pmatrix} m \\ p \end{pmatrix},$$

where

$$M(m, p, \epsilon) = \begin{pmatrix} 1 & \epsilon a_{23} m_3 & \epsilon a_{23} m_2 & 0 & \epsilon b_{23} p_3 & \epsilon b_{23} p_2 \\ \epsilon a_{31} m_3 & 1 & \epsilon a_{31} m_1 & \epsilon b_{31} p_3 & 0 & \epsilon b_{31} p_1 \\ \epsilon a_{12} m_2 & \epsilon a_{12} m_1 & 1 & \epsilon b_{12} p_2 & \epsilon b_{12} p_1 & 0 \\ 0 & \epsilon a_2 p_3 & -\epsilon a_3 p_2 & 1 & -\epsilon a_3 m_3 & \epsilon a_2 m_2 \\ -\epsilon a_1 p_3 & 0 & \epsilon a_3 p_1 & \epsilon a_3 m_3 & 1 & -\epsilon a_1 m_1 \\ \epsilon a_1 p_2 & -\epsilon a_2 p_1 & 0 & -\epsilon a_2 m_2 & \epsilon a_1 m_1 & 1 \end{pmatrix}.$$

This map will be denoted dGC in what follows.

A “simple” integral of the map dGC can be obtained by discretizing the following Wronskian relation with constant coefficients, which holds for the general flow of the Clebsch system (84):

$$A_1(\dot{m}_1 p_1 - m_1 \dot{p}_1) + A_2(\dot{m}_2 p_2 - m_2 \dot{p}_2) + A_3(\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0,$$

with

$$A_i = a_i a_j + a_i a_k - a_j a_k.$$

Proposition 53. *The set $\Gamma = (\tilde{m}_1 p_1 - m_1 \tilde{p}_1, \tilde{m}_2 p_2 - m_2 \tilde{p}_2, \tilde{m}_3 p_3 - m_3 \tilde{p}_3)$ is a HK basis for the map dGC with $\dim K_\Gamma(x) = 1$. At each point $x \in \mathbb{R}^6$ we have: $K_\Gamma(x) = [e_1 : e_2 : e_3]$, where, for $(i, j, k) = \text{c.p.}(1, 2, 3)$,*

$$e_i = A_i + \epsilon^2 a_i (b_i - b_j) A_k \Theta_j + \epsilon^2 a_i (b_i - b_k) A_j \Theta_k,$$

with

$$\Theta_i = p_i^2 + \frac{a_i}{\theta a_j a_k} m_i^2.$$

(Recall that θ is defined by eq. (82); we assume here that $\theta \neq \infty$.)

As in the case of the first flow, the integrals e_i/e_j can be expressed through one symmetric integral: $e_i/e_j = (A_i - \theta a_i L)/(A_j - \theta a_j L)$, where

$$L = \frac{a_2 a_3 A_1 \Theta_1 + a_3 a_1 A_2 \Theta_2 + a_1 a_2 A_3 \Theta_3}{1 + \epsilon^2 \theta a_1 a_2 a_3 (\Theta_1 + \Theta_2 + \Theta_3)}.$$

The quantities e_i and the integral L can be also obtained from a different (monomial) HK basis, given in part b) of the following Proposition.

Proposition 54.

(a) *The set $\Phi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1, m_2 p_2, m_3 p_3, 1)$ is a HK basis for the map dGC with $\dim K_\Phi(m, p) = 4$. Thus, any orbit of the map dGC lies on an intersection of four quadrics in \mathbb{R}^6 .*

(b) *The set of functions $\Phi_0 = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, 1)$ is a HK basis for the map dGC with $\dim K_{\Phi_0}(m, p) = 1$. At each point $(m, p) \in \mathbb{R}^6$ there holds:*

$$K_{\Phi_0}(m, p) = [a_2 a_3 e_1 : a_3 a_1 e_2 : a_1 a_2 e_3 : (a_1/\theta) e_1 : (a_2/\theta) e_2 : (a_3/\theta) e_3 : -e_0],$$

where

$$e_0 = a_2 a_3 A_1 \Theta_1 + a_3 a_1 A_2 \Theta_2 + a_1 a_2 A_3 \Theta_3$$

is an integral of motion of the continuous time flow (84).

(c) *The sets of functions (87)–(89) are HK bases for the map dGC with one-dimensional null-spaces.*

(d) *Each of the sets of functions $\Psi_0 = (\tilde{p}_1 p_1, \tilde{p}_2 p_2, \tilde{p}_3 p_3, \tilde{m}_1 m_1, \tilde{m}_2 m_2, \tilde{m}_3 m_3, 1)$ and (91)–(93) is a HK basis for the map dGC with a one-dimensional null-space.*

15. $\mathfrak{su}(2)$ RATIONAL GAUDIN SYSTEM WITH $N = 2$ SPINS

The Gaudin system [7] describes an interaction of N quantum spins y_i , $i = 1, \dots, N$, with a homogeneous constant external field p . Its classical version is given by the following quadratic system of differential equations [19]:

$$\dot{y}_i = \left(\lambda_i p + \sum_{j=1}^N y_j \right) \times y_i, \quad 1 \leq i \leq N, \quad (95)$$

where $y_i \in \mathfrak{su}(2) \simeq \mathbb{R}^3$, $p \in \mathfrak{su}(2) \simeq \mathbb{R}^3$ is a constant vector, and pairwise distinct numbers λ_i are parameters of the model. The flow (95) is Hamiltonian with respect to the Lie-Poisson bracket of the direct sum of N copies of $\mathfrak{su}(2)$, admits $2N$ independent conserved quantities in involution: the N Casimir functions

$$C_k = \langle y_k, y_k \rangle,$$

and the following N Hamiltonians:

$$H_k = \langle p, y_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle y_k, y_j \rangle}{\lambda_k - \lambda_j},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathfrak{su}(2) \simeq \mathbb{R}^3$. Note that the Hamilton function of the flow (95) is

$$H = \sum_{k=1}^N \lambda_k H_k = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \langle y_i, y_j \rangle + \sum_{i=1}^N \lambda_i \langle p, y_i \rangle.$$

In [15, 19] it has been proved that a contraction of N simple poles to one pole of order N provides the integrable flow of the so called *one-body rational $\mathfrak{su}(2)$ tower*, whose simplest instance with $N = 2$ describes the dynamics of the three-dimensional Lagrange top in the rest frame.

We consider here the HK discretization of the flow (95) with $N = 2$. We set $y_1 = (x_1, x_2, x_3)^T$, $y_2 = (z_1, z_2, z_3)^T$, and choose the constant gravity vector $p = (0, 0, 1)^T$. We thus obtain the following system of differential equations:

$$\begin{cases} \dot{x}_1 = x_2 z_3 - x_3 z_2 + \lambda_1 x_2, \\ \dot{x}_2 = x_3 z_1 - x_1 z_3 - \lambda_1 x_1, \\ \dot{x}_3 = x_1 z_2 - x_2 z_1, \\ \dot{z}_1 = z_2 x_3 - z_3 x_2 + \lambda_2 z_2, \\ \dot{z}_2 = z_3 x_1 - z_1 x_3 - \lambda_2 z_1, \\ \dot{z}_3 = z_1 x_2 - z_2 x_1, \end{cases} \quad (96)$$

with λ_1, λ_2 being real parameters. The system (96) has the following four independent integrals of motion:

$$\begin{aligned} C_1 &= x_2^2 + x_2^2 + x_3^2, & C_2 &= z_1^2 + z_2^2 + z_3^2, \\ H_1 &= x_3 + \frac{x_1 z_1 + x_2 z_2 + x_3 z_3}{\lambda_1 - \lambda_2}, & H_2 &= z_3 + \frac{x_1 z_1 + x_2 z_2 + x_3 z_3}{\lambda_2 - \lambda_1}. \end{aligned}$$

Note that the quantity $H_1 + H_2 = x_3 + z_3$ is a linear integral of motion. We mention also the following Wronskian relation with constant coefficients:

$$(x_3 + z_3)(\dot{x}_1 z_1 - x_1 \dot{z}_1 + \dot{x}_2 z_2 - x_2 \dot{z}_2) + (\lambda_1 + \lambda_2 + x_3 + z_3)(\dot{x}_3 z_3 - x_3 \dot{z}_3) = 0. \quad (97)$$

The HK discretization of (96) reads:

$$\begin{cases} \tilde{x}_1 - x_1 = \epsilon(\tilde{x}_2 z_3 + x_2 \tilde{z}_3 - \tilde{x}_3 z_2 - x_3 \tilde{z}_2) + \epsilon \lambda_1(\tilde{x}_2 + x_2), \\ \tilde{x}_2 - x_2 = \epsilon(\tilde{x}_3 z_1 + x_3 \tilde{z}_1 - \tilde{x}_1 z_3 - x_1 \tilde{z}_3) - \epsilon \lambda_1(\tilde{x}_1 + x_1), \\ \tilde{x}_3 - x_3 = \epsilon(\tilde{x}_1 z_2 + x_1 \tilde{z}_2 - \tilde{x}_2 z_1 - x_2 \tilde{z}_1), \\ \tilde{z}_1 - z_1 = \epsilon(\tilde{z}_2 x_3 + x_2 \tilde{z}_3 - \tilde{z}_3 x_2 - z_3 \tilde{x}_2) + \epsilon \lambda_2(\tilde{z}_2 + z_2), \\ \tilde{z}_2 - z_2 = \epsilon(\tilde{z}_3 x_1 + x_3 \tilde{z}_1 - \tilde{z}_1 x_3 - z_1 \tilde{x}_3) - \epsilon \lambda_2(\tilde{z}_1 + z_1), \\ \tilde{z}_3 - z_3 = \epsilon(\tilde{z}_1 x_2 + x_1 \tilde{z}_2 - \tilde{z}_2 x_1 - z_2 \tilde{x}_1). \end{cases} \quad (98)$$

The map $f : x \mapsto \tilde{x}$ obtained by solving (98) for \tilde{x} is given by:

$$\tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(\mathbb{1} + \epsilon B)x,$$

where $x = (x_1, x_2, x_3, z_1, z_2, z_3)^T$, and

$$A(x, \epsilon) = \begin{pmatrix} 1 & -\epsilon z_3 & \epsilon z_2 & 0 & \epsilon x_3 & -\epsilon x_2 \\ -\epsilon z_3 & 1 & -\epsilon z_1 & -\epsilon x_3 & 0 & \epsilon x_1 \\ -\epsilon z_2 & \epsilon z_1 & 1 & \epsilon x_2 & -\epsilon x_1 & 0 \\ 0 & \epsilon z_3 & -\epsilon z_2 & 1 & -\epsilon x_3 & \epsilon x_2 \\ -\epsilon z_3 & 0 & \epsilon z_1 & \epsilon x_3 & 1 & -\epsilon x_1 \\ \epsilon z_2 & -\epsilon z_1 & 0 & -\epsilon x_2 & \epsilon x_1 & 1 \end{pmatrix} - \epsilon B,$$

$$B = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This map will be called dG in the sequel. The quantity $x_3 + z_3$ is obviously preserved by the map dG. Other conserved quantities may now be found using the HK bases approach. A “simple” integral follows, as in the previous sections, by discretizing the Wronskian relation (97).

Proposition 55. *The set of functions $\Gamma = (\tilde{x}_1 z_1 - x_1 \tilde{z}_1, \tilde{x}_2 z_2 - x_2 \tilde{z}_2, \tilde{x}_3 z_3 - x_3 \tilde{z}_3)$ is a HK basis for the map dG with the one-dimensional null-space $K_\Gamma(x, z) = [x_3 + z_3 : x_3 + z_3 : I]$, with*

$$I(x, z) = \frac{\lambda_1 + \lambda_2 + x_3 + z_3 + \epsilon^2 \lambda_1(x_1^2 + x_2^2) + \epsilon^2 \lambda_2(z_1^2 + z_2^2) + \epsilon^2(\lambda_1 + \lambda_2)(x_1 z_1 + x_2 z_2)}{1 - \epsilon^2(\lambda_1 x_3 + \lambda_2 z_3 + \lambda_1 \lambda_2)}.$$

A full set of integrals is found in the following Proposition. The roles of the variables x_i and z_i are not quite symmetric there, and interchanging them is of course admissible but does not lead to new integrals of motion.

Proposition 56.

(a) *The set $\Phi = (x_1^2 + x_2^2, z_1^2 + z_2^2, z_3^2, x_1 z_1 + x_2 z_2, z_3, 1)$ is a HK basis for the map dG with $\dim K_\Phi(x) = 3$.*

(b) The set $\Phi_1 = (1, z_3, z_3^2, x_1^2 + x_2^2)$ is a HK basis for the map dG with a one-dimensional null-space. At each point $(x, z) \in \mathbb{R}^6$ we have: $K_{\Phi_1}(x, z) = [c_0 : c_1 : c_2 : -1]$. The functions c_0, c_1, c_2 are conserved quantities of the map dG , given by

$$\begin{aligned} c_0 &= \frac{x_1^2 + x_2^2 - 2x_3z_3 - z_3^2 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{\Delta_1 \Delta_2}, \\ c_1 &= \frac{2(x_3 + z_3) \left(1 + \epsilon^2 c_1^{(3)} + \epsilon^4 c_1^{(5)} + \epsilon^6 c_1^{(7)} + \epsilon^8 c_1^{(9)}\right)}{\Delta_1 \Delta_2}, \\ c_2 &= -\frac{\left(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)} + \epsilon^8 c_2^{(10)}\right)}{\Delta_1 \Delta_2}, \end{aligned}$$

where

$$\Delta_1 = 1 - \epsilon^2(\lambda_1 x_3 + \lambda_2 z_3 + \lambda_1 \lambda_2), \quad \Delta_2 = 1 + \epsilon^2 \Delta_2^{(2)} + \epsilon^4 \Delta_2^{(4)} + \epsilon^6 \Delta_2^{(6)}.$$

Here $\Delta^{(q)}$ and $c_k^{(q)}$ are polynomials of degree q in the phase variables. In particular:

$$c_2^{(2)} = -(x_1^2 + x_2^2 + x_3^2) - (z_1^2 + z_2^2 + z_3^2) - 2(x_1 z_1 + x_2 z_2 + x_3 z_3) - 2(\lambda_2 x_3 + \lambda_1 z_3) - (\lambda_1^2 + \lambda_2^2),$$

$$\text{and } \Delta_2^{(2)} = c_2^{(2)} + \lambda_1 x_3 + \lambda_2 z_3 + \lambda_1 \lambda_2.$$

(c) The set $\Phi_2 = (1, z_3, z_3^2, x_1 z_1 + x_2 z_2)$ is a HK basis for the map dG with a one-dimensional null-space. At each point $(x, z) \in \mathbb{R}^6$ we have: $K_{\Phi_2}(x, z) = [d_0 : d_1 : d_2 : -1]$. The functions d_0, d_1, d_2 are conserved quantities of the map dG , given by

$$\begin{aligned} d_0 &= \frac{x_1 z_1 + x_2 z_2 + x_3 z_3 + (\lambda_2 - \lambda_1) z_3 + \epsilon^2 d_0^{(4)} + \epsilon^4 d_0^{(6)} + \epsilon^6 d_0^{(8)} + \epsilon^8 d_0^{(10)}}{\Delta_1 \Delta_2}, \\ d_1 &= \frac{\lambda_1 - \lambda_2 - x_3 - z_3 + \epsilon^2 d_1^{(3)} + \epsilon^4 d_1^{(5)} + \epsilon^6 d_1^{(7)} + \epsilon^8 d_1^{(9)}}{\Delta_1 \Delta_2}, \\ d_2 &= -\frac{1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)} + \epsilon^8 c_2^{(8)}}{\Delta_1 \Delta_2}, \end{aligned}$$

where $d_k^{(q)}$ are polynomials of degree q in the phase variables.

(d) The set $\Phi_3 = (1, z_3, z_3^2, z_1^2 + z_2^2)$ is a HK basis for the map dG with a one-dimensional null-space. At each point $(x, z) \in \mathbb{R}^6$ we have: $K_{\Phi_3}(x, z) = [e_0 : e_1 : e_2 : -1]$. The functions e_0, e_1, e_2 are conserved quantities of the map dG , given by

$$\begin{aligned} e_0 &= \frac{z_1^2 + z_2^2 + z_3^2 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1 \Delta_2}, \\ e_1 &= \frac{2\epsilon^2 \left(e_1^{(2)} + \epsilon^2 e_1^{(4)} + \epsilon^4 e_1^{(6)} + \epsilon^6 e_1^{(8)}\right)}{\Delta_1 \Delta_2}, \\ e_2 &= -\frac{1 + \epsilon^2 e_2^{(2)} + \epsilon^4 e_2^{(4)} + \epsilon^6 e_2^{(6)} + \epsilon^8 e_2^{(8)}}{\Delta_1 \Delta_2}, \end{aligned}$$

where $e_k^{(q)}$ are polynomials of degree q in the phase variables.

It can be shown that each of the sets $\{c_0, c_1, c_2, x_3 + z_3\}$, $\{d_0, d_1, d_2, x_3 + z_3\}$ and $\{e_0, e_1, e_2, x_3 + z_3\}$ contains four independent integrals.

Analogously to the situation for the map dLT, it is possible to obtain a polynomial integral and an invariant volume form for the map dG.

Proposition 57. *The function*

$$G = \frac{1}{2}(x_1 + z_1)^2 + \frac{1}{2}(x_2 + z_2)^2 + \lambda_1 x_3 + \lambda_2 z_3 - \frac{\epsilon^2}{2}((\lambda_1 x_1 + \lambda_2 z_1)^2 + (\lambda_1 x_2 + \lambda_2 z_2)^2),$$

is a conserved quantity for the map dG.

Proposition 58. *The map dG possesses an invariant volume form:*

$$\det \frac{\partial(\tilde{x}, \tilde{z})}{\partial(x, z)} = \frac{\phi(\tilde{x}, \tilde{z})}{\phi(x, z)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dz_1 \wedge dz_2 \wedge dz_3}{\phi(x, z)},$$

where $\phi(x, z) = \Delta_2(x, z)$.

Explicit integration of the map dG could be based on the following claim.

Proposition 59. *The component x_3 of the solution of the dG map satisfies a relation of the type*

$$Q(x_3, \tilde{x}_3) = q_0 x_3^2 \tilde{x}_3^2 + q_1 x_3 \tilde{x}_3 (x_3 + \tilde{x}_3) + q_2 (x_3^2 + \tilde{x}_3^2) + q_3 x_3 \tilde{x}_3 + q_4 (x_3 + \tilde{x}_3) + q_5 = 0,$$

coefficients of the biquadratic polynomial Q being conserved quantities of dG. Thus, $x_3(t)$ is an elliptic function of degree 2. An analogous statement holds for the component z_3 .

16. CONCLUSIONS

The initial motivation for this study was the hope that HK discretization would preserve the integrability for all algebraically integrable systems. This was formulated as a conjecture in [17]. The list of integrable discretizations given in the present overview contains more than a dozen issues and is rather impressive. It includes systems integrable in terms of elliptic functions as well as those integrable in terms of theta-functions of genus $g = 2$ (Clebsch system). This list might look like a convincing argument in favor of the integrability conjecture. However, at present we have also found examples which indicate that this conjecture might be wrong (e.g., the Zhukovski-Volterra system with all $\beta_k \neq 0$, or integrable chains, Volterra and dressing ones, with a big number of particles, say $N \geq 5$). We do not have rigorous proofs of the non-integrability in these cases, but the numerical evidence is rather strong. Since HK discretizations are (probably) not always integrable, the big number of integrable cases becomes still more intriguing: it is hard to imagine that all their common features come as a pure coincidence. We are after a theory which would clarify the problem of integrability of HK discretizations, but at present such a theory seems to remain a rather remote goal. Still, even without a general framework, HK discretizations represent a new fascinating chapter in the theory of integrable systems: we are now in a possession of a big and potentially growing stock of birational maps, integrable in terms of Abelian functions, highly non-trivial from the point of view of algebraic geometry and very different in nature from anything known before. The immediate goal of this review will be achieved if HK discretizations will attract attention of experts in the theory of integrable systems and in algebraic geometry.

REFERENCES

- [1] M. Adler, P. van Moerbeke, P. Vanhaecke, *Algebraic Integrability, Painlevé Geometry and Lie Algebras*, Springer, Berlin, 2004.
- [2] M. Alber, G.G. Luther, J.E. Marsden, J. Robbins, “Geometric phases, reduction and Lie-Poisson structure for the resonant three-wave interaction”, *Phys. D* 123 (1998), 271–290.
- [3] I. Basak, “Explicit solution of the Zhukovski-Volterra gyrostat”, *Reg. Chaot. Dyn.* 14:2 (2009), 223–236.
- [4] A. Clebsch, “Über die Bewegung eines Körpers in einer Flüssigkeit”, *Math. Ann.* 3 (1870), 238–262.
- [5] V. Dragovic, B. Gajic, “Hirota-Kimura type discretization of the classical nonholonomic Suslov problem”, *Reg. Chaot. Dyn.* 13:4 (2008), 250–256.
- [6] F. Golse, A. Mahalov, B. Nicolaenko, “Bursting dynamics of the 3D Euler equations in cylindrical domains, instability in models connected with fluid flows”, *Int. Math. Series* 6:7 (2008), 300–338.
- [7] M. Gaudin, “Diagonalisation d’une classe d’Hamiltoniens de spin”, *Jour. Phys.* 37 (1976), 1089–1098.
- [8] R. Hirota and K. Kimura, “Discretization of the Euler top”, *Jour. Phys. Soc. Jap.* 69:3 (2000), 627–630.
- [9] N.J. Hitchin, N.S. Manton, M.K. Murray, “Symmetric monopoles”, *Nonlin.* 8:5 (1995), 661–692.
- [10] W. Kahan, “Unconventional numerical methods for trajectory calculations”, *Unpublished lecture notes*, 1993.
- [11] W. Kahan and R.-C. Li, “Unconventional schemes for a class of ordinary differential equations – with applications to the Korteweg-de Vries equation”, *Jour. Comp. Phys.* 134 (1997), 316–331.
- [12] K. Kimura and R. Hirota, “Discretization of the Lagrange top”, *Jour. Phys. Soc. Jap.* 69:10 (2000), 3193–3199.
- [13] G. Kirchhoff, “Über die Bewegung eines Rotationskörpers in einer Flüssigkeit”, *J. Reine Angew. Math.* 71 (1870), 237–262.
- [14] J. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics 17, 2nd edition, New York: Springer, 1999.
- [15] F. Musso, M. Petrera, O. Ragnisco and G. Satta, “A rigid body dynamics derived from a class of extended Gaudin models: an integrable discretization”, *Regul. Chaot. Dyn.* 10 (2005), 363–380.
- [16] A.M. Perelomov, *Integrable Systems of Classical Mechanics and Lie algebras*, Basel: Birkhäuser, 1990.
- [17] M. Petrera, A. Pfadler and Yu.B. Suris, “On integrability of Hirota-Kimura-type discretizations: experimental study of the discrete Clebsch system”, *Exp. Math.* 18:2 (2009), 223–247.
- [18] M. Petrera and Yu.B. Suris, “On the Hamiltonian structure of Hirota-Kimura discretization of the Euler top”, *Math. Nachr.* (to appear), e-print: [arXiv:0707.4382 \[math-ph\]](https://arxiv.org/abs/0707.4382).
- [19] M. Petrera and Yu.B. Suris, “An integrable discretization of the rational $\mathfrak{su}(2)$ Gaudin model and related systems”, *Comm. Math. Phys.* 283 (2008), 227–253.
- [20] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, “Integrable mappings and soliton equations”, *Phys. D* 34 (1989), 183–192.
- [21] T. Ratiu, “Nonabelian semidirect product orbits and their relation to integrable systems”, *Oberwolfach Reports* 3, Nr. 1, 2006.
- [22] A.G. Reyman and M.A. Semenov-Tian-Shansky, “Group theoretical methods in the theory of finite dimensional integrable systems” – In: *Encyclopaedia of mathematical science, Vol. 16: Dynamical Systems VII*. Berlin: Springer, 1994, 116–225.
- [23] J.M. Sanz-Serna, “An unconventional symplectic integrator of W. Kahan”, *Appl. Numer. Math.* 16 (1994), 245–250.
- [24] Yu.B. Suris, “On integrable standard-like mappings”, *Funct. Anal. Appl.* 23 (1989), 74–76.
- [25] Yu.B. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*, Progress in Mathematics 219, Basel: Birkhäuser, 2003.
- [26] G. Suslov, *Theoretical mechanics*, Gostekhizdat, Moscow-Leningrad, 1946.
- [27] A.P. Veselov, A.B. Shabat, “Dressing chains and the spectral theory of the Schrödinger operator”, *Funct. Anal. Appl.* 27:2 (1993), 81–96.
- [28] V. Volterra, “Sur la théorie des variations des latitudes”, *Acta Math.* 22 (1899), 201–357.